METHODS OF APPROXIMATE INTEGRATION

by

GEORGE D. MOUAT, B.Sc. (Glas.)

The Association of Engineering and Shipbuilding Draughtsmen

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PREFACE.

THE object of this pamphlet is to gather within two covers all the methods and rules of approximate integration which are used by designers and draughtsmen at some time or other in their work. Many calculations can be made, using integrals which give exact results or alternatively lead to exact formulae, but there are other calculations which have to be made where the variables do not follow any mathematical law. For instance, the cross sections of a reservoir could be very irregular, and in this case the volume can only be determined by some method of approximate integration.

It is assumed that the reader will have some knowledge of the differential and integral calculus, sufficient at least to understand the meaning of the symbols used, and while it is a simple matter to use the various rules for approximately finding the area of a figure, the user will have much more confidence and know the best rule to use in different circumstances if he fully understands its derivation and limitations.

The approximate rules can be obtained in several ways; the simplest in each case has been adopted, and more rigorous derivations are not required.

The examples are selected for their general interest, and taken to illustrate as far as possible the type of calculations made every day in many drawing offices.

SYNOPSIS OF CONTENTS.

Introduction—Nature of Curves—Moment Curves—Integral Curves—The Parabola—Simpson's 1st Rule—Fractional Spacings—A Useful Mnemonic—Accuracy of 1st Rule, with worked example—Area, Centroid and 2nd Moment Calculation—Thomson's Rule—Simpson's 2nd Rule—Calculation of Centroid and 2nd Moment about Base Line—The Five Eight minus One Rule—Tchebycheff's Rules—Calculation of Radius of Gyration of solid of revolution—Rules by Gauss—Derivation of a New Rule—Application to Ship Calculations—Worked-out comparisons with Simpson's and Tchebycheff's Rules. Summary—Examples using the New Rule—Graphic Integration—Example using time-velocity diagram—Polar Co-ordinates—Radian Measure—Polar Integration—Areas and Centroid of Sectors, with examples of Calculations—Mechanical Integration—Description and Theory of the Planimeter and the Integrator—The Integraph.

METHODS OF APPROXIMATE INTEGRATION

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Introduction.

If y is a function of x, a statement which is written

$$y = F(x)$$

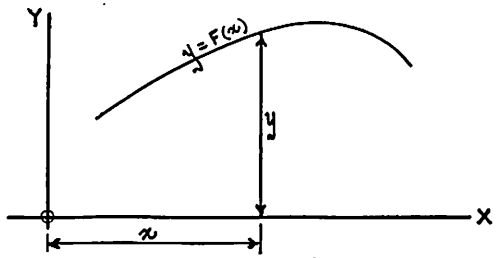

Fig. 1.

then the relation between x and y can be seen visually by calculating the ordinates y at various abscissae x and drawing the graph as in Fig. 1.

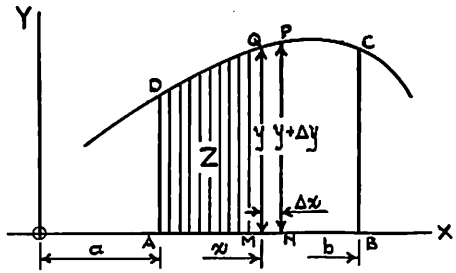

Fig. 2.

If now in Fig. 2 the area under the curve between A and B is required, this could be obtained by dividing AB into a great number of strips of width $\triangle x$. The area of each strip would be $y \,.\, \triangle x$ plus a small triangular portion at the top, and the total area would be the sum of the areas of all such strips.

The area Z, which is shaded, may be regarded as being swept out by the ordinate MQ in moving from AD. To every value of x

there is a value of Z ; therefore Z is a function of x. If x increases by $\triangle x$, then Z increases by MNPQ $= \triangle Z$ and from figure :—

$\triangle Z$ is greater than $y \triangle x$ and less than $(y + \triangle y) \triangle x$.
$\triangle Z / \triangle x$,, ,, y ,, ,, $y + \triangle y$.

Now $\triangle Z / \triangle x$, when $\triangle Z$ and $\triangle x$ are considered to become infinitely small, tends to a definite ratio which is written dZ/dx, and since, at the same time, $\triangle y$ also becomes infinitely small and eventually zero, then

$$dZ/dx = y \tag{1}$$

or, stated in words, the rate of change of area (Z) with respect to abscissae (x) is the ordinate (y). The same statement, since y is equal to F (x) is

$$dZ/dx = \mathrm{F}\ (x)$$

or, the differential coefficient of Z with respect to x is F (x).

Let the anti-differential, or what is more usually known as the integral of F (x), be ϕ (x), then, the sign of integration being $\int$ which is the old-fashioned elongated s, the first letter in the word summation :

$$\begin{aligned} \mathrm{Z} &= \int \mathrm{F}\ (x)\ dx \\ &= \phi\ (x) + \mathrm{C} \end{aligned} \tag{2}$$

where C is a constant, added to make the statement perfectly general, since the differential coefficient of a constant is zero.

Now, when x is equal to a, Z is equal to zero, therefore

$$\begin{aligned} \phi\ (a) + \mathrm{C} &= 0 \\ \mathrm{C} &= -\phi\ (a) \end{aligned}$$

whence $\quad \mathrm{Z} = \phi\ (x) - \phi\ (a)$

This is the area between the fixed ordinate $x = a$ and the variable ordinate $x = x$. Hence the area between the ordinates AD and BC is $\phi\ (b) - \phi\ (a)$ by putting $x = b$. In the notation of the calculus this is written

$$\begin{aligned} \phi\ (b) - \phi\ (a) &= \Big[\ \phi\ (x)\ \Big]_{x=a}^{x=b} \\ &= \Big[\ \int \mathrm{F}\ (x)\ dx\ \Big]_{x=a}^{x=b} \\ &= \int_a^b \mathrm{F}\ (x)\ dx \end{aligned} \tag{3}$$

Here the *range* of integration is from a to b, and (3) is known as a *definite integral*, whereas (2) is called an indefinite integral.

It has now been shown that an area can be represented by an integral. The converse is also true, that an integral can be represented by an area, and it does not matter whether the integral is a volume, a mass, or other quantity, as the number of area units in the figure equals the number of units in the integral.

It follows from the proceeding work that if $y = F(x)$ is known and that the anti-differential $\phi(x)$ of $F(x)$ can be found, then it is a simple matter to find the area between any desired values of a and b. It often happens that $F(x)$ is unknown, and is extremely troublesome to find, therefore the anti-differential is unknown, and other methods must be found to find the area under a curve.

Nature of Curves.

The curves usually dealt with can be classified under three heads :—

(1) Mathematical.—These are the curves of definite functions, which are so important that they are given names such as parabola, cycloid, hyperbola, astroid, etc.

(2) Regular.—Also known as "smooth" or "fair." These are the curves drawn for many purposes, a good example being the lines of a ship, where great care is taken to ensure that the curves will not be irregular.

(3) Capricious.—This type of curve is well illustrated by the contour lines of a hill, or even by a steam engine indicator diagram notwithstanding the hyperbolic basis of the latter.

Integrals in the first group would usually, but not always, be found exactly by the processes of the integral calculus. The integral of a^{x^2} is a case where a mathematical function must be integrated by approximate methods, as this function has no known integral. In the second group the areas under the curves would be found by some approximate rule, and the third group would be best integrated by the planimeter, or even the integrator if more than the area, for instance, moments were required.

Moment Curves.

In Fig. 2 the area of the element MNPQ is $y \triangle x$, where $\triangle x$ is very small, and the 1st moment of this element about the OY axis is $x\, y\, \triangle x$.

If the function of the curve is known, then the total moment of the area could be evaluated by the integral :

$$M_1 = \int_a^b x \,.\, F(x)\, dx \qquad (1)$$

With arbitrary curves the function is unknown, and the integral is evaluated by other means.

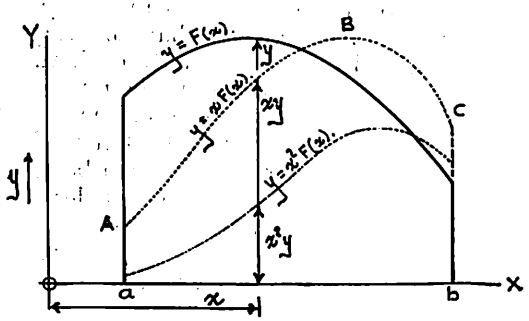

Fig. 3.

In Fig. 3 values of xy have been calculated, and these values set up to a convenient scale at the appropriate abscissae, as ordinates for a new curve ABC. This curve represents $x\,\mathrm{F}(x)$, and the area under it is equivalent to the 1st moment of the original curve about OY.

In a similar manner, the 2nd moment is represented by the integral :

$$\mathrm{M}_2 = \int_a^b x^2\ \mathrm{F}\ (x)\ dx.* \qquad (2)$$

Hence, by setting up values of x^2y as ordinates, drawing the curve, and evaluating the area under it, the 2nd moment about OY could be found.

The curves $x\ \mathrm{F}\ (x)$ and $x^2\ \mathrm{F}\ (x)$ do not require to be actually drawn when calculating the 1st and 2nd moments, and they are only shown in Fig. 3 for illustration. Nevertheless there are times when they *should* be drawn in, in order that the shape may give an indication to the best rule to use for their integrations.

Integral Curves.

In Fig. 4 the area shaded has been shown to be :

$$\mathrm{Z} = \int_a^x \mathrm{F}\ (x)\ dx$$

$$= \Big[\ \phi\ (x)\ \Big]_{x=a}^{x=x}$$

$$= \phi\ (x) - \phi\ (a)$$

When x is equal to a, the area Z is zero. If, now, values of Z are calculated for various values of x, the values of Z could be used as ordinates of a new curve AVW drawn to a suitable scale. Any

* When mass is being dealt with, the 2nd moment becomes the Moment of Inertia. This term will not be used unless mass is involved.

ordinate of the curve AVW represents graphically the area, and therefore is the integral of the original curve up to that ordinate, and because of this property the curve AVW is called the integral curve of the original or fundamental curve.

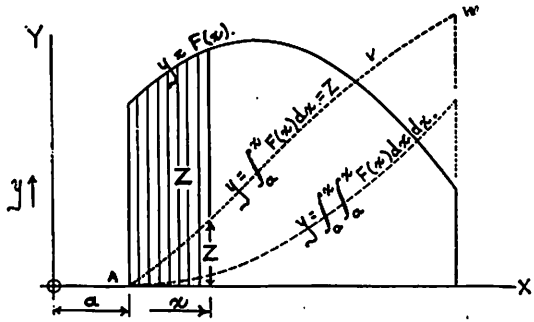

Fig. 4.

The integral curve has itself an integral curve, which is called the second integral curve of the fundamental curve, and it is evident that the fundamental curve could have any number of successive integral curves constructed from it. These curves are of great use in solving practical problems in mechanics, statics and physics. The properties of integral curves do not, however, come within the scope of this work.

The Parabola.

When a function of x is of the form :

$$y = ax^n + bx^{n-1} + \ldots\ldots\ldots + \beta x + \alpha$$

where a, b, β, α are constants, and n is an integer, the graph of this function is called a parabola of the nth order. All rules of approximate integration are based on functions having parabolic curves of various orders. The rules can now be enumerated.

Simpson's First Rule.

Thomas Simpson (1710-1761) was the professor of mathematics at Woolwich. This rule, which is also known as the ONE-THIRD RULE, was, however, published by James Stirling (1690-1772) in his *"Methodus Differentialis"* in 1730.

The rule is derived by assuming the curve to be a parabola of the 2nd order, in which case the function is :

$$y = ax^2 + bx + c.$$

and contains three constants, which is one more than the order of the parabola. It follows that if three ordinates of the curve and their abscissae are known, it would be possible to define the function, but such definition is seldom, if ever, required in approximate integration. Three ordinates are, however, necessary in finding the area approximately.

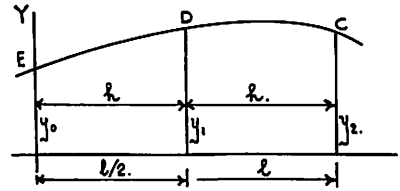

Fig. 5.

In Fig. 5 the area by definite integral is :

$$Z = \int_0^l (ax^2 + bx + c)\, dx$$

$$= a\, l^3/3 + b\, l^2/2 + c\, l \qquad (1)$$

Let the ordinates spaced $l/2$ apart be $y_0 y_1$ and y_2 and the area Z be $my_0 + ny_1 + py_2$ where m, n and p are multipliers to be determined.

Now, at $x = 0$ $\quad y_0 = c$

$x = l/2$ $\quad y_1 = c + bl/2 + a\, l_2/4$

$x = l$ $\quad y_2 = c + bl + a\, l^2$

$$Z = mc + n\{c + bl/2 + al^2/4\} + p\{c + bl + al^2\}$$

$$= c\{m + n + p\} + bl\{n/2 + p\} + al^2\{n/4 + p\} \qquad (2)$$

The two equations (1) and (2) must be equal, and on equating terms which contain the same constants :

$$m + n + p = l$$

$$n/2 + p = l/2 \quad \Big\} \quad n/4 = l/6$$

$$n/4 + p = l/3 \quad \Big\} \quad n = 2\, l/3$$

$$p = l/3 - n/4 = l/3 - l/6 = l/6$$

$$m = l - n - p = l - 2\, l/3 - l/6 = l/6$$

Hence the rule is :

$$\text{Area} = l\, y_0/6 + 2\, l\, y_1/3 + l\, y_2/6$$

or as it is more usually written

$$\{y_0 + 4y_1 + y_2\}\, h/3$$

where h is the "common interval" between the ordinates, and is equal to $l/2$.

In words the rule is—To the first and third ordinates, add four times the second ordinate, and multiply this sum by one-third of the distance between the ordinates, to find the area.

The first rule, although derived using a parabolic curve of the 2nd order, exactly integrates one of the third order,* for if

$$y = ax^3 + bx^2 + cx + d$$

then the area from $x = 0$ to $x = 2h$, is

$$\int_0^{2h} y \cdot dx = 4\, a\, h^4 + 8\, b\, h^3/3 + 2\, ch^2 + 2\, dh \qquad (1)$$

* This can only be regarded as a coincidence.

The ordinates at $x = 0$, h, and $2h$ are :

At $x = 0$	$y_0 = d$
$x = h$	$y_1 = d + ch + bh^2 + ah^3$
$x = 2h$	$y_2 = d + 2ch + 4bh^2 + 8ah^3$

The area using the first rule is :

$$\{y_0 + 4y_1 + y_2\}\, h/3 = \{6d + 6ch + 8bh^2 + 12ah^3\}\, h/3$$
$$= 2dh + 2ch^2 + 8bh^3/3 + 4ah^4$$

which agrees with (1) above. The importance of this last proof lies in the fact that if a curve is assumed to be a second order parabola, then the same rule will correctly integrate its moment curve, which is of the third order, and leads to a very useful rule for finding the centroid of a curvilinear figure. In Fig. 6, the ordinates h apart are y_0, y_1, y_2 ; the corresponding ordinates, which would be required in calculating the moment about the y_0 ordinate are $y_0 \times 0$, $y_1 \times h$, $y_2 \times 2h$.

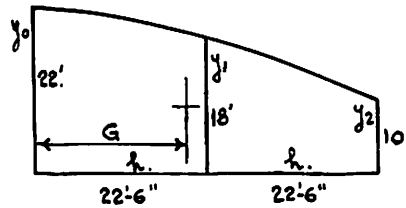

Fig. 6.

$$\text{therefore } G = \frac{\{y_0 \times 0 + 4y_1 \times h + y_2 \times 2h\}\, h/3}{\{\, y_0 + 4y_1 + y_2 \,\}\, h/3}$$

$$= \frac{4y_1 + 2y_2}{y_0 + 4y_1 + y_2}\, h$$

In a certain figure, $y_0 = 22'$ $y_1 = 18'$ $y_2 = 10'$ and the distance between these ordinates is 22'-6". Figure 6 is drawn to these dimensions and

$$G = 22{\cdot}5\ \{72 + 20\} / \{22 + 72 + 10\}$$
$$= 19{\cdot}9 \text{ ft.}$$

The first rule is extended to cater for long figures, such as a ship's waterplane, by dividing the length into an even number of sections (n), each of length h, as in figure 7. The ordinate lengths are y_0, y_1, y_2, etc., to y_{n-1}, y_n.

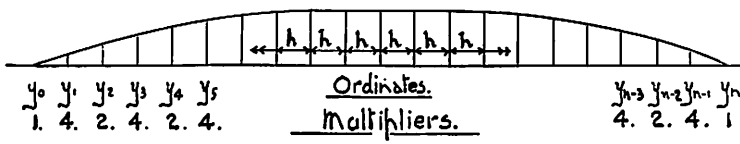

Fig. 7.

The end ordinates y_0 and y_n may be zero, and the area of the half waterplane is :—

$$[y_0+4y_1+y_2]\ h/3 + [y_2+4y_3+y_4]\ h/3 + \ldots\ldots$$
$$\ldots\ldots + [y_{n-2} + 4y_{n-1} + y_n]\ h/3$$
$$= [y_0+4y_1+2y_2+4y_3+ \ldots\ldots + 2y_{n-2}+4y_{n-1}+y_n]\ h/3.$$

The area is sometimes taken to be :

$$[y_0/2 + 2y_1 + y_2 + \ldots\ldots + y_{n-2} + 2y_{n-1} + y_n/2]\ 2h/3$$

and this considerably lessens the arithmetical work. In the figure, it will be seen that part of the outline of the waterplane is parallel to the centre line for some distance. Greater accuracy in calculating areas will generally be obtained by using more ordinates where the curvature is greatest. These additional ordinates are obtained :

(*a*) by dividing two end sections of length h into two parts of length $h/2$, as in Fig. 8, where the area is :

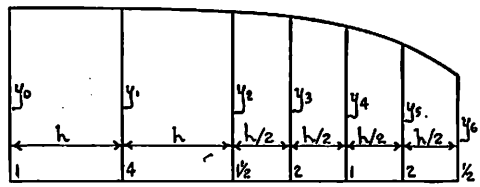

Fig. 8.

$$A = \{y_0 + 4y_1 + y_2\}\ h/3 + \{y_2 + 4y_3 + y_4\}\ h/3.2$$
$$+ \{y_4 + 4y_5 + y_6\}\ h/3.2$$
$$= \{y_0 + 4y_1 + 3y_2/2 + 2y_3 + y_4 + 2y_5 + y_6/2\}\ h/3$$

the multipliers being 1, 4, 3/2, 2, 1, 2, 1/2.

(*b*) by using sections which are fractional parts of h, a method so useful that it should be better known.

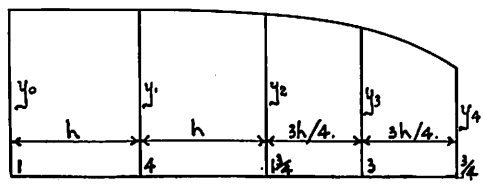

Fig. 9.

In Fig. 9 the area is :

$$A = \{y_0 + 4y_1 + y_2\}\ h/3 + \{y_2 + 4y_3 + y_4\}\ 3\ h/4.3$$
$$= \{y_0 + 4y_1 + 7y_2/4 + 3y_3 + 3y_4/4\}\ h/3$$

the multipliers being 1 . 4 . 7/4 . 3 . 3/4.

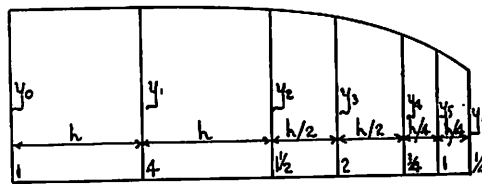

Fig. 10.

In cases where the curvature is very great, or for other reasons, the sections can be taken to give ordinates $h/4$ apart as in Fig. 10, where the area is :

$$A = \{y_0 + 4y_1 + y_2\}\ h/3 + \{y_2 + 4y_3 + y_4\}\ h/2.3 + \{y_4 + 4y_5 + y_6\}\ h/4.3$$

$$= \{y_0 + 4y_1 + 3y_2/2 + 2y_3 + 3y_4/4 + y_5 + y_6/4\}\ h/3$$

the multipliers being 1 . 4 . 3/2 . 2 . 3/4 . 1 . 1/4.

The length of the figure is $7h/2 = L$, giving $h = 2L/7$, and the base is divided in the manner shown.

Figures 11 and 12 have been drawn to show at a glance the various fractional spacing and multipliers that can be adopted to suit various conditions. These should be used whenever the required calculations justify their adoption. The thick ordinates shown in these diagrams is an indication that, if moments are to be calculated, one of these ordinates must be taken as an axis.

In ship calculations the waterplane in Fig. 7 for a ship 500 feet or over, would usually be calculated using 21 ordinates. Greater accuracy with less arithmetical work would be obtained by using ordinates spaced as in Fig. 13, where (*c*) is for a full ship with considerable parallel middle body, and (b) is for a finer form. At a) the spacing of the normal 21 ordinates is shown.

In these calculations the centroid (centre of flotation) and the 2nd moment of the area would be required. These items are calculated about the midship ordinate as axis, and in order to illustrate the curves which would be integrated in the calculations, the curves are drawn in, in dotted lines in Fig. 13. It will be seen that the ordinates are very well placed for performing the integrations, being at the positions of greatest curvature, and this explains the statement that greater accuracy will be obtained by using fractional spacings. The actual number of ordinates is less, and the multipliers no more difficult to use, therefore the amount of arithmetical work is reduced.

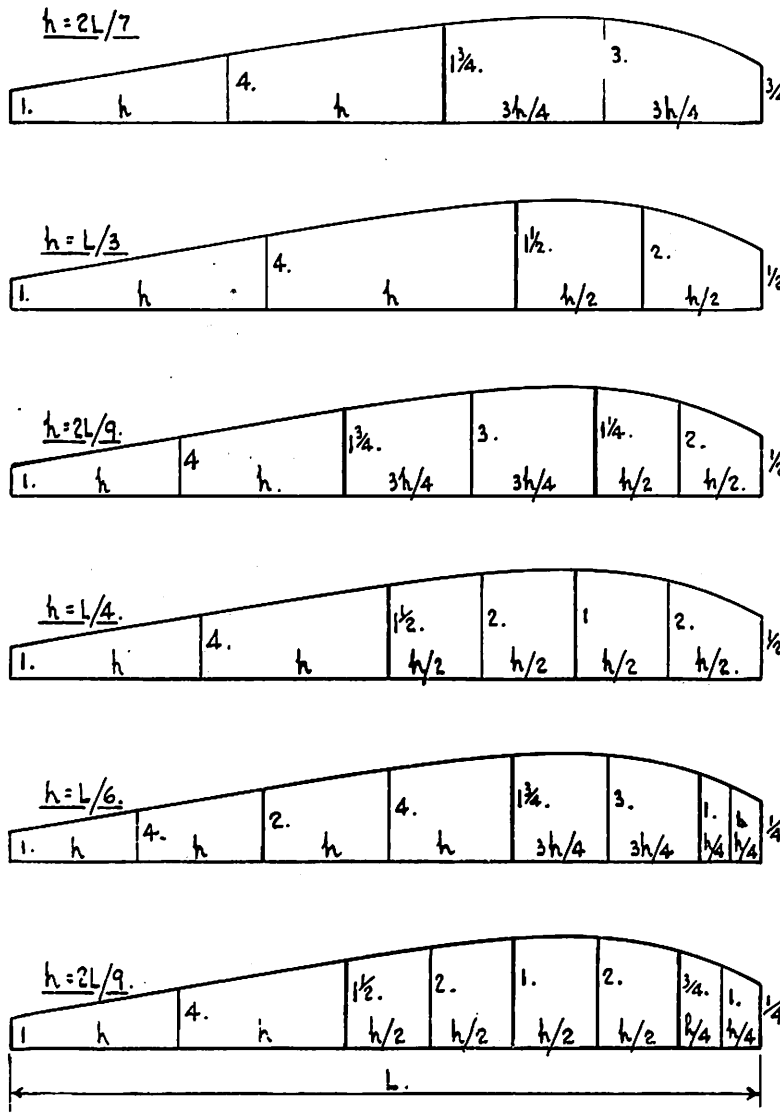
h = 2L/7
1. h
4. h
1¾. 3h/4
3. 3h/4
¾.
h = L/3
1. h
4. h
1½. h/2
2. h/2
½.
h = 2L/9.
1. h
4 h.
1¾. 3h/4
3. 3h/4
1¼. h/2
2. h/2.
½
h = L/4.
1. h
4. h
1½. h/2
2. h/2
1 h/2
2. h/2.
½.
h = L/6.
1. h
4. h
2. h
4. h
1¾. 3h/4
3. 3h/4
1. h/4
h h/4
¼.
h = 2L/9.
1 h
4. h
1½. h/2
2. h/2
1. h/2
2. h/2
¾. h/4
1. h/4
¼
L.

Fig. 11.

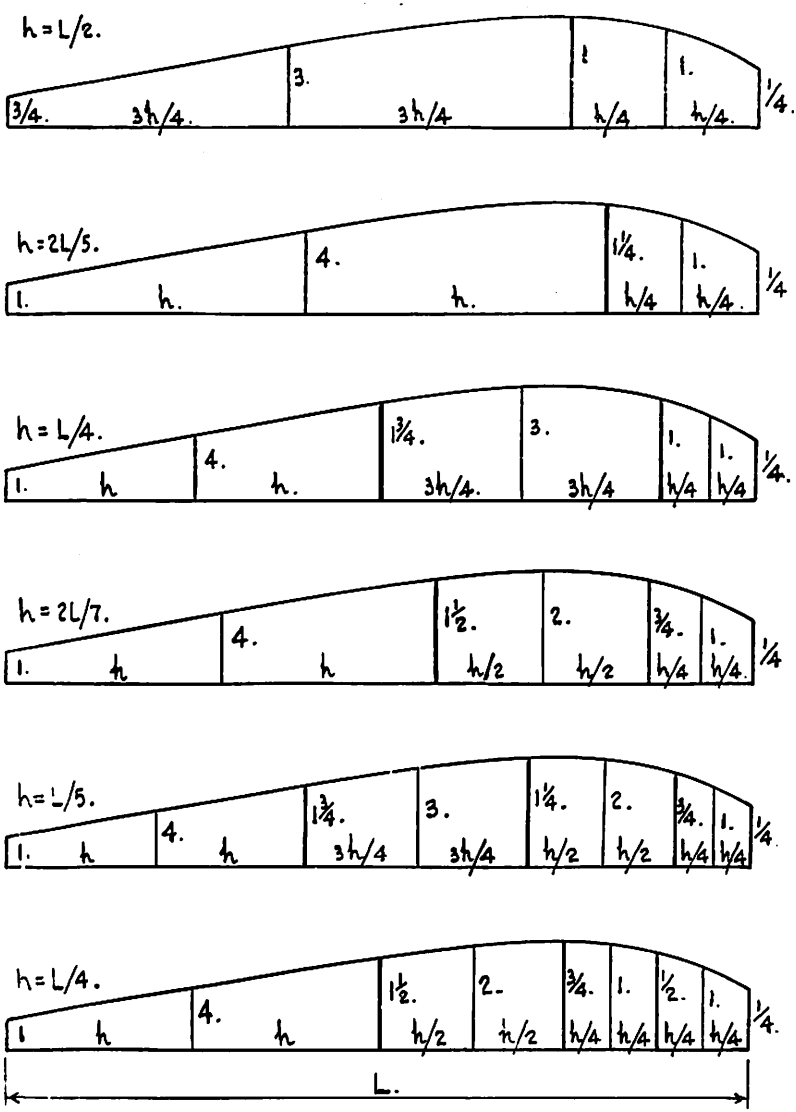
h = L/2.
3/4. 3h/4. 3. 3h/4 2 h/4 1. h/4. 1/4.
h = 2L/5.
1. h. 4. h. 1¼. h/4 1. h/4. 1/4
h = L/4.
1. h 4. h. 1¾. 3h/4. 3. 3h/4 1. h/4 1. h/4 1/4.
h = 2L/7.
1. h 4. h 1½. h/2 2. h/2 ¾. h/4 1. h/4. 1/4
h = L/5.
1. h 4. h 1¾. 3h/4 3. 3h/4 1¼. h/2 2. h/2 ¾. h/4 1. h/4 1/4.
h = L/4.
1 h 4. h 1½. h/2 2. h/2 ¾. h/4 1. h/4 ½. h/4 1. h/4 1/4.
L.

Fig. 12.

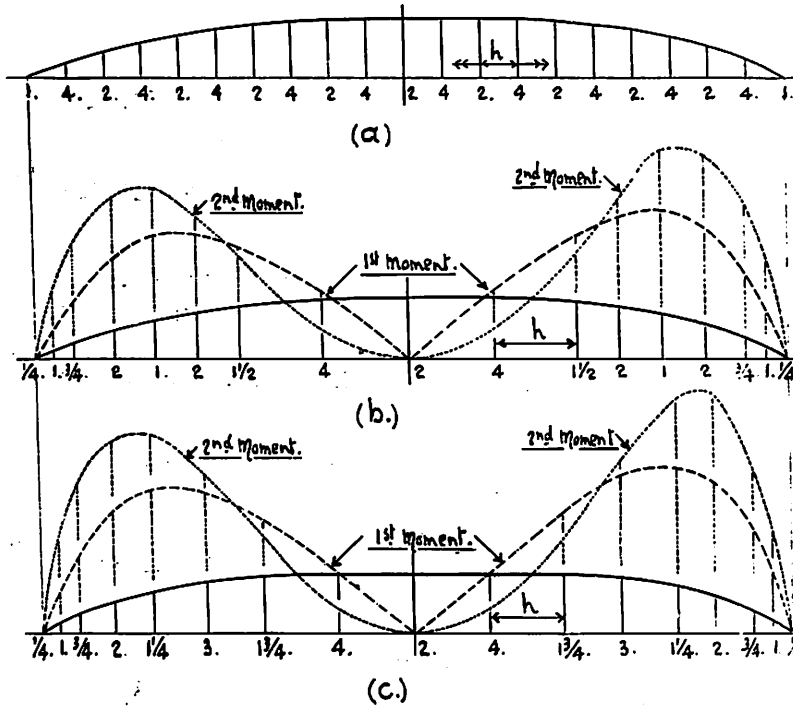
h
1. 4. 2. 4. 2. 4 2 4 2 4 2 4 2. 4 2 4 2. 4 2 4. 1.
(a)
2nd Moment.
2nd Moment.
1st Moment.
h
¼. 1.¾. 2 1. 2 1½ 4 2 4 1½ 2 1 2 ¾ 1.¼.
(b.)
2nd Moment.
2nd Moment.
1st Moment.
h
¼. 1.¾. 2. 1¼ 3. 1¾. 4. 2. 4. 1¾. 3. 1¼. 2. ¾. 1 ¼
(c.)

Fig. 13.

A Useful Mnemonic.

In arriving at the 1st rule, the curve assumed was $y = ax^2 + bx + c$, a parabola of the second order. The rule will always integrate a parabolic curve of this order exactly, for instance, the plan sections of a frustrum of a cone or pyramid vary as the square of the distance from the apex, so that the volume of these frustra could be integrated without error by the rule. Further, when x becomes very great the curve of the function approximates to a straight line, and therefore the rule correctly integrates the area under a straight line. This fact may be used as a mnemonic, for in Fig. 14, the area is $3h$ where the height is unity, and by rule the area is :

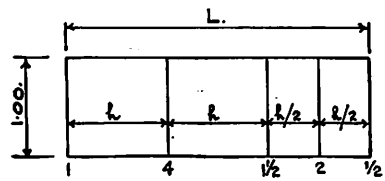

Fig. 14.

$$[1 + 4 + 3/2 + 2 + 1/2]\ h/3 = 3\ h = L$$

and the reader can confirm that in all cases.

[Sum of multipliers] $h/3$ = Length of figure.

In particular, the area of the first diagram in Fig. 12 is :

$$[3y_0/4 + 3y_1 + y_2 + y_3 + y_4/4]\ h/3.$$

If the ordinates are of unit height :

$$[3/4 + 3 + 1 + 1 + 1/4]\ h/3 = 2\,h = L.$$

This variation of the rule must be used carefully to avoid taking $3\,h/4$ as the common interval.

Accuracy of the 1st Rule.

The curves usually dealt with are regular or fair curves, and it is assumed that between certain sections these approximate to parabolas of the second order. The question of accuracy arises, and this can only be investigated by actual example. For this purpose the curve of the function $y = \log_{10} x$ has been selected, because this curve has entirely different properties to those of the parabola, and because the ordinates are easily obtained from a table of logarithms. The following particulars will be calculated, for the range between $x = 1$ and $x = 15$:

(i.) The Area,
(ii.) The 1st Moment,
(iii.) The position of the Centroid,
(iv.) The 2nd Moment about the Centroid,

by calculus methods, and compared with results obtained using the rule. With definite integrals the axes are necessarily always the reference axes for moments. When moments are being integrated by approximate rules, the axes should always be chosen as near to the centroid of the figure as possible, in order that the arithmetical work be as small, numerically and quantitively, as possible.

The function $y = \text{Log}_{10} x$ can only be integrated by transferring to the hyperbolic form

$$y = \text{Log}_e x / \text{Log}_e\ 10$$

The denominator, $\text{Log}_e\ 10$ in this expression is a constant, equal to 2·3026, and therefore since $1/2{\cdot}3026 = 0{\cdot}4343$.

$$(1)\ \text{Area} = 0{\cdot}4343 \int_1^{15} \text{Log}_e x = 0{\cdot}4343 \Big[x\ \text{Log}_e x - x \Big]_1^{15} \quad *$$

$$= 11{\cdot}5617 \text{ sq. units.}$$

$$(2)\ \text{1st Moment} = 0{\cdot}4343 \int_1^{15} x\ \text{Log}_e x = 0{\cdot}4343 \Big[(x^2\ \text{Log}_e x)/2 - x^2/4 \Big]_1^{15}$$

$$= 107{\cdot}9936 \text{ units.}^3$$

* Note :—The General Integral

$$\int x^n\ \text{Log}_e\ x = [x^{n+1}\ \text{Log}_e\ x]/(n+1) - x^{n+1}/(n+1)^2 + c$$

The reader can check these results as an exercise, remembering that hyperbolic logarithms must be used in the calculations.

(3) The centroid from the OY axis is

1st moment/area

$= 107{\cdot}99 \text{ units}^3/11{\cdot}56 \text{ units}^2 = 9{\cdot}342 \text{ units}.$

(4) 2nd Moment $= 0{\cdot}4343\int_1^{15} x^2 \operatorname{Log}_e x = 0{\cdot}4343\left[(x^3 \operatorname{Log}_e x)/3 - x^3/9\right]_1^{15}$

$= 1160{\cdot}3 \text{ units}.^4$

It is proved in text books on physics, that if I_{GG} is the 2nd moment of a figure about its centroid, then the 2nd moment about any other axis YY which is parallel to GG is

$$I_{YY} = I_{GG} + Ad^2,$$

where A is the area of the figure and d the perpendicular distance between the axes.

Hence $$I_{GG} = I_{YY} - Ad^2$$

and, in the case being dealt with,

$A = 11{\cdot}5617$, $d = 9{\cdot}342$, giving

$I_{GG} = 1160{\cdot}3 - 1008{\cdot}7$

$= 151{\cdot}6 \text{ units}.^4$

These particulars will now be calculated, using Simpson's 1st rule.

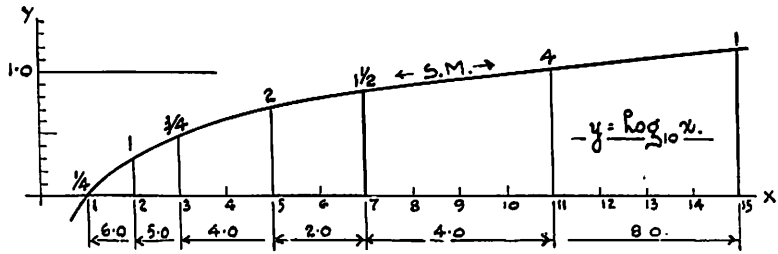

Fig. 15.

The function $y = \operatorname{Log}_{10} x$ is shown in Fig. 15. It will be seen that the curvature is greatest at the left-hand side. The ordinates to be used are shown, and the ordinate at $x = 7$ will be taken as the reference axis for moments. The area is then partly on each side of this axis, and clearly, the 1st moment preponderates to the right of the axis. The 2nd moment is always positive since squares are never negative and, therefore, is the sum of these moments on each side of the axis chosen. The calculation is best arranged in tabular form methodically :

Ordinates.	S M	Products for Area.	Lever	Products for 1st Mt.	Lever	Products for 2nd Mt.
Log 1 = 0·0	½	0·0	−6	0·0	−6	0·0
,, 2 = 0·30103	1	0·30103	−5	−1·50515	−5	7·52575
,, 3 = 0·47712	¾	0·35784	−4	−1·43136	−4	5·72544
,, 5 = 0·69897	2	1·39794	−2	−2·79588	−2	5·59176
,, 7 = 0·84510	1½	1·26765	0	−5·73239	0	0·0
,, 11 = 1·04139	4	4·16556	4	16·66224	4	66·64896
,, 15 = 1·17613	1	1·17613	8	9·40904	8	75·27232
		Σ = 8·66615		Σ = 26·07128		Σ = 160·76423

(1) Area = 8·66615 × 4/3 = 11·5549 sq. units.

(2) 1st Moment = {26·07128 − 5·73239} × 4/3
= 27·11852 units.^{3}

(3) Position of Centroid = 27·1182/11·5549
= 2·34 units to right of ordinate at $x = 7$

(4) 2nd Moment = 160·76423 × 4/3
= 214·35231 units.^{4}

The 2nd Moment about an axis through the Centroid is
$214·4 - 11·55 \times 2·34^2 = 151·1$ units.^{4}

These figures compare very favourably with those calculated by definite integrals, and show that when sufficient ordinates are taken, the rule can be used with confidence.

Thomson's Rule.

This rule is derived from Simpson's 1st rule as follows :

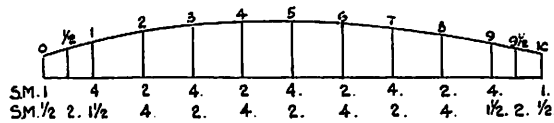

Fig. 16.

In Fig. 16 the length is divided into an even number of sections, say 10, with half sections in the first and last. Then, neglecting the actual lengths of the ordinates, and calling the area A :—

Section	0	½	1	2	3	4	5	6	7	8	9	9½	10	
A = [	1		4	2	4	2	4	2	4	2	4		1]	$h/3$
A = [	½	2	1½	4	2	4	2	4	2	4	1½	2	½]	$h/3$
2A = [	1½	2	5½	6	6	6	6	6	6	6	5½	2	1½]	$h/3$ By Addition
A = [	¼	⅜	11/12	1	1	1	1	1	1	1	11/12	⅜	¼]	h By Division.

The use of this rule may reduce labour in some cases, due to many of the multipliers being unity.

Simpson's 2nd Rule.

This rule assumes that the area to be integrated is bounded by a function of the form $y = ax^3 + bx^2 + cx + d$.

The expression contains four constants a, b, c and d. Therefore, if four ordinates are known, they could be used to find the values of these constants, the equation of the curve, and the area by definite integral. This, as in the 1st rule, is not actually done.

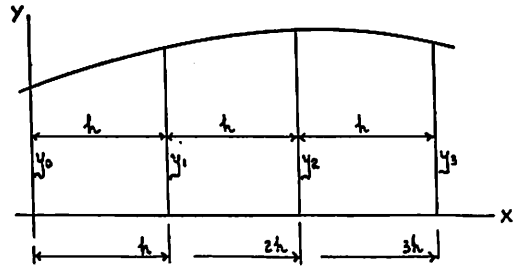

Fig. 17.

In figure 17, the area under the curve between the ordinates at $x = 0$ and $x = 3h$ is :

$$A = \int_0^{3h} (ax^3 + bx^2 + cx + d)\, dx$$

$$= 81\ a\ h^4/4 + 27\ b\ h^3/3 + 9\ c\ h^2/2 + 3\ d\ h \qquad (1)$$

It is required that the area will be given by the form :

$$A = m y_0 + n y_1 + p y_2 + q y_3$$

where m, n, p and q are the multipliers of the ordinates to which they are attached.

Now

$$y_0 = d$$
$$y_1 = ah^3 + bh^2 + ch + d$$
$$y_2 = 8ah^3 + 4bh^2 + 2ch + d$$
$$y_3 = 27ah^3 + 9bh^2 + 3ch + d$$

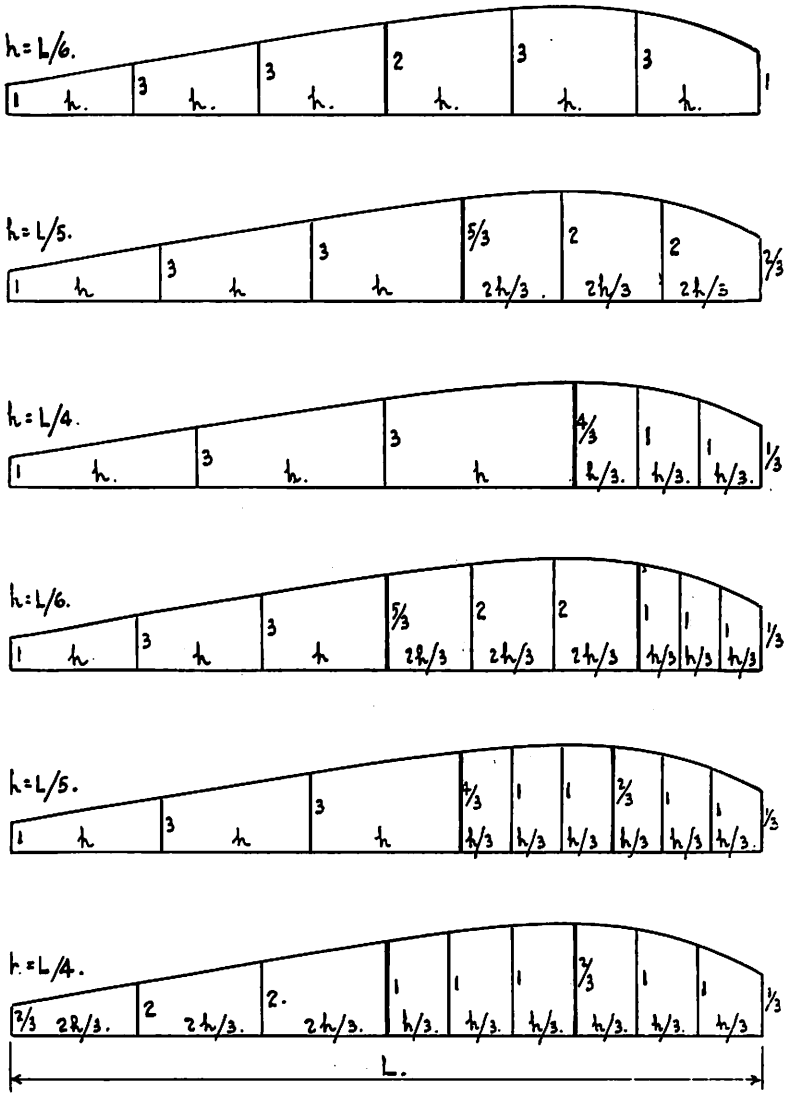
h = L/6.
1 h. 3 h. 3 h. 2 h. 3 h. 3 h. 1
h = L/5.
1 h 3 h 3 h 5/3 2h/3. 2 2h/3 2 2h/3 2/3
h = L/4.
1 h. 3 h. 3 h 4/3 h/3. 1 h/3. 1 h/3. 1/3
h = L/6.
1 h 3 h 3 h 5/3 2h/3 2 2h/3 2 2h/3 1 h/3 1 h/3 1 h/3 1/3
h = L/5.
1 h 3 h 3 h 4/3 h/3 1 h/3 1 h/3 2/3 h/3 1 h/3 1 h/3 1/3
h = L/4.
2/3 2h/3. 2 2h/3. 2. 2h/3. 1 h/3. 1 h/3. 1 h/3. 2/3 h/3. 1 h/3. 1 h/3 1/3
L.

Fig. 18.

Therefore $A = d\,(m + n + p + q) + ch\,(n + 2p + 3q) + bh^2\,(n + 4p + 9q) + ah^3\,(n + 8p + 27q)$ (2)

In order to eliminate the constants, corresponding terms in (1) and (2) are equated, giving :

$$\left.\begin{aligned} m + n + p + q &= 3h \\ n + 2p + 3q &= 9h/2 \\ n + 4p + 9q &= 9h \\ n + 8p + 27q &= 81h/4 \end{aligned}\right\} \qquad \begin{aligned} 4p + 12q &= 9h \\ 4p + 18q &= 45h/4 \\ \hline 6q &= 9h/4 \\ q &= 3h/8 \end{aligned}$$

$$4p = 9h - 9h/2 = 9h/2$$
$$p = 9h/8$$
$$n = 9h - 36h/8 - 27h/8 = 9h/8.$$
$$m = 3h - 9h/8 - 9h/8 - 3h/8 = 3h/8.$$

The required rule is therefore :

$$A = 3h\,y_0/8 + 9\,hy_1/8 + 9h\,y_2/8 + 3h\,y_3/8,$$

or as it is more usually written :

$$A = \{y_0 + 3y_1 + 3y_2 + y_3\}\ 3h/8,$$

the ordinate multipliers being 1 . 3 . 3 . 1.

The rule does not have the facility of the first rule, in integrating the curve of the next higher order parabola, that is the 4th, exactly, but the difference is so slight that this does not matter in practical calculations, where the curves would not, usually, be even parabolic.

The second rule can be used with fractional spacings, and these are shown in Fig. 18. In suitable cases the employment of these variations leads to decreased labour, but the spacing in the last diagram should be used carefully, remembering that the area is :

$$\{{}^2/_3 . 2 . 2 . 1 . 1 . 1 . {}^2/_3 . 1 . 1 . {}^1/_3\}\ 3h/8,$$

and that there is no section of length h in the figure.

The mnemonic :

{ Sum of multipliers } $3h/8$ = Length,

does, however, apply.

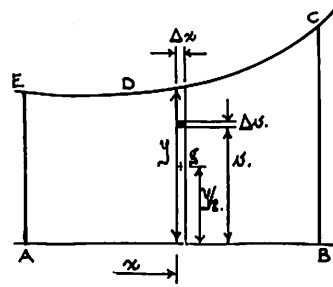

Fig. 19.

A calculation of frequent occurrence is that of finding the 1st and 2nd moments, about AB as axis, of the area ABCDE in Fig 19.

The theoretical basis of the calculation lies in taking a strip of breadth $\triangle x$. The ordinate is y, and the area $y \triangle x$, $\triangle x$ being small. The centroid is at $y/2$ above AB, and therefore the 1st moment of the strip is $y^2 \triangle x/2$. Hence if $y^2/2$ is set up at the corresponding ordinates, and a curve drawn to some scale, the area under this curve represents the 1st moment. The 2nd moment of the element $\triangle x \,.\, \triangle v$ about AB is $\triangle x \triangle v\, v^2$, and the 2nd moment of the entire strip is :

$$\triangle x \int_0^y v^2 d\, v = \triangle x\, y^3/3.$$

The 2nd moment for the whole area ABCDE could, therefore, be found by drawing a new curve having ordinates $y^3/3$, the area of which represents the 2nd moment.

An area taken to illustrate the manner of calculating the 1st and 2nd moments, (*a*) about the left-hand ordinate, and (*b*) about the base line by Simpson's 2nd rule, is shown in Fig. 20. The tabulated calculation is below the figure along with explanatory notes.

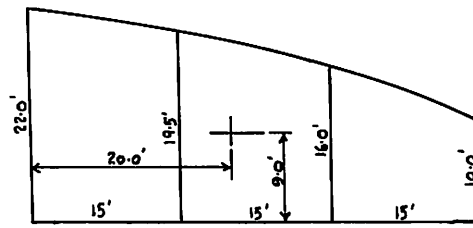

Fig. 20.

Ord.	Length	M	F (Area)	Lever	F (1st Mt.)	Lever	F (2nd Mt.)
1	22·0	1	22·0	0	0·0	0	0·0
2	19·5	3	58·5	1	58·5	1	58·5
3	16·0	3	48·0	2	96·0	2	192·0
4	10·0	1	10·0	3	30·0	3	90·0
(*a*) Summations→			138·5		184·5		340·5

(i) Area = $138{\cdot}5 \times 15 \times \frac{3}{8} = 779{\cdot}6$ sq. ft.

(ii) 1st Moment = $(184{\cdot}5 \times 15) \times 15 \times \frac{3}{8} = 15567$ ft.^{3} units.

(iii) Position of Centroid = $184{\cdot}5 \times 15/138{\cdot}5 = 19{\cdot}98$ ft.

(iv) 2nd Moment = $(340{\cdot}5 \times 15^2) \times 15 \times \frac{3}{8} = 430940$ ft.^{4} units

Ord.	Length	Length^2	Length^3	M	F (1st Mt.)	F (2nd Mt.)
1	22·0	484	10,648	1	484	10,648
2	19·5	380	7,415	3	1,140	22,245
3	16·0	256	4,096	3	768	12,288
4	10·0	100	1,000	1	100	1,000

(*b*) Summations→ 2,492 46,181

(v) 1st Moment = $(\frac{1}{2} \times 2492) \times 15 \times \frac{3}{8} = 70087$ ft.^{3} units.

(vi) Position of Centroid = $2492/2 \times 138{\cdot}5 = 8{\cdot}99$ ft.

(vii) 2nd Moment = $(\frac{1}{3} \times 46181) \times 15 \times \frac{3}{8} = 86588$ ft.^{4} units.

(i) The Common Interval is 15.

(ii) 184·5 is multiplied by 15 because the "levers" in the table have been taken 1, 2, 3 instead of 15, 30, 45.

(iii) $15 \times \frac{3}{8}$ has been cancelled in numerator and denominator.

(iv) 340·5 is multiplied by 15^2 because the squares of the levers are 15^2, 30^2, 45^2 and not 1^2, 2^2, 3^2.

(v) 2492 is multiplied by $\frac{1}{2}$, because the integral being evaluated is $\frac{1}{2} \int y^2 dx$.

(vi) $15 \times \frac{3}{8}$ has been cancelled in numerator and denominator.

(vii) 46181 is multiplied by $\frac{1}{3}$ because the integral being evaluated is $\frac{1}{3} \int y^3 dx$.

Rules of approximate integration have been derived using up to 14 ordinates. The first two rules, that is, the rules for three and four ordinates are the only ones of practical use. As a matter of interest, the multipliers when the ordinates number up to eight, are given in the following table. The common interval is h.

No. of Ordinates.	External Factor.	Ordinate Multipliers.
3	$2h/6$	1 . 4 . 1
4	$3h/8$	1 . 3 . 3 . 1
5	$4h/90$	7 . 32 . 12 . 32 . 7
6	$5h/288$	19 . 75 . 50 . 50 . 75 . 19
7	$6h/840$	41 . 216 . 27 . 272 . 27 . 216 . 41
8	$7h/17280$	751 . 3557 . 1323 . 2989 . 2989 . 1323 . 3557 . 751

Beyond this number of ordinates the multipliers become even more unwieldy, and in some cases negative.

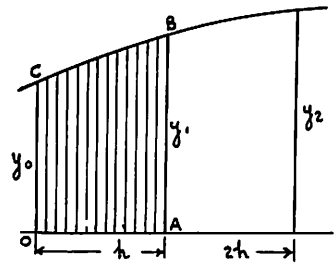

Fig. 21.

The Five, Eight, minus One Rule.

This useful rule has been devised to obtain the area OABC, Fig. 21, given the ordinates y_0 y_1 and y_2.

The equation of the curve is taken to be $y = ax^2 + bx + c$. Hence the area by definite integral is :

$$\int_0^h y\,dx = ah^3/3 + bh^2/2 + ch \qquad (1)$$

and using multipliers m, n, and p, the area will be $m\,y_0 + ny_1 + py_2$.

When $x = 0$ $\quad y_0 = c$

$x = h$ $\quad y_1 = ah^2 + bh + c$

$x = 2h$ $\quad y_2 = 4\,ah^2 + 2\,bh + c$

The area is therefore equal to

$$m\,c + n\{c + bh + ah^2\} + p\{c + 2\,bh + 4\,ah^2\}$$
$$= (m + n + p)\,c + bh\,(n + 2p) + ah^2\,(n + 4p) \qquad (2)$$

On equating corresponding terms in (1) and (2).

$$m + n + p = h$$
$$n + 2p = h/2 \quad \Big\} \quad 2p = h/3 - h/2$$
$$n + 4p = h/3 \quad \Big\} \quad p = -h/12.$$
$$n = h/3 - 4p = h/3 + 4h/12 = 8h/12.$$
$$m = h - n - p = h - 8h/12 + h/12 = 5h/12.$$

and the rule is :

$$\text{Area} = (5\,y_0 + 8y_1 - y_2)\,h/12.$$

The multipliers 5, 8, -1, explain the name given to the rule, which does not correctly integrate the 1st moment curve when OC is the axis, for the equation of the moment curve is

$$y = ax^3 + bx^2 + cx \qquad (1)$$

and the first moment up to $x = h$ is :

$$\int_0^h y\,dx = ah^4/4 + bh^3/3 + ch^2/2 \qquad (2)$$

In (1) when $x = 0$ $\quad y_0 = 0$

$x = h$ $\quad y_1 = ah^3 + bh^2 + ch$

$x = 2h$ $\quad y_2 = 8\,ah^3 + 4\,bh^2 + 2\,ch.$

Using the 5, 8, -1 rule, the 1st moment would be :

$$\text{M} = (5 \times 0 + 8\,ah^3 + 8\,bh^2 + 8\,ch - 8\,ah^3 - 4\,bh^2 - 2\,ch)\,h/12$$
$$= bh^3/3 + ch^2/2,$$

which is short of equation (2) by the quantity $ah^4/4$, and is therefore incorrect.

The first moment may, however, be calculated, using the fundamental curve with a new set of ordinate multipliers, say u, v and w, in which case :

$$\text{M} = u\,c + v\,(ah^2 + bh + c) + w\,(4\,ah^2 + 2\,bh + c)$$
$$= c\,(u + v + w) + bh\,(v + 2w) + ah^2\,(v + 4w) \qquad (3)$$

On equating corresponding terms in (2) and (3)

$$u + v + w = h^2/2$$
$$v + 2w = h^2/3$$
$$v + 4w = h^2/4$$

from which, $u = 3h^2/24$, $v = 10h^2/24$, $w = -h^2/24$, and the 1st moment may, therefore, be calculated by the formula :

$$M = (3y_0 + 10y_1 - y_2)\ h^2/24$$

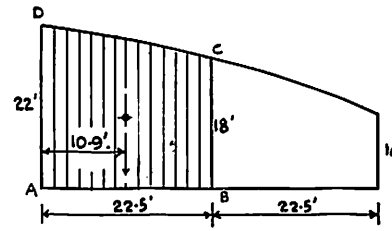

Fig. 22.

In Fig. 22, the centroid of ABCD from AD is therefore :—

$$G = \frac{(3y_0 + 10y_1 - y_2)\ h^2/24}{(5y_0 + 8y_1 - y_2)\ h/12}$$

and where $y_0 = 22'$, $y_1 = 18'$, $y_2 = 10'$ and $h = 22'\text{-}6''$, the calculation would be performed in tabular form :

Ord.	Multiplier.	Products for Area.	Multiplier.	Products for 1st Moment.
22·0	5	110	3	66
18·0	8	144	10	180
		254		246
10·0	−1	−10	−1	−10
		$\Sigma = 244$		$\Sigma = 236$

Area $= 244 \times 22{\cdot}5/12.$

1st Moment $= 236 \times 22{\cdot}5 \times 22{\cdot}5/24.$

Position of Centroid $= 236 \times 22{\cdot}5/244 \times 2 = 10{\cdot}9$ ft.

Tchebycheff's Rules.

The Russian shipbuilder of this name devised a number of rules for determining the area of a curvilinear figure, in which the lengths of the ordinates suitably spaced, only require *adding together* to obtain the area. Now, in an nth order parabola, if all the ordinates

have a unit multiplier, then the number of ordinates to be used must be n. Thus in a 2nd order parabola, two ordinates are required.

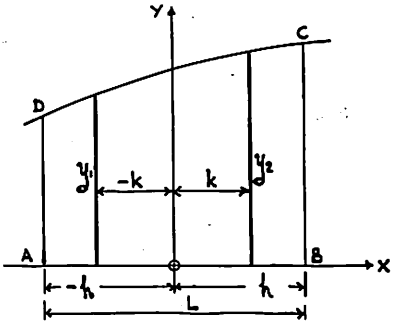

Fig. 23.

In Fig. 23 the origin is taken at the centre of base, and the area ABCD by definite integral is :

$$\int_{-h}^{h} y\,dx = \int_{-h}^{h} (ax^2 + bx + c)\,dx$$

$$= \left[ax^3/3 + bx^2/2 + cx\right]_{-h}^{h}$$

$$= 2\,ah^3/3 + 2\,ch \qquad (1)$$

Let the two ordinates be spaced $\pm k$ about the origin. Then the area required

$= (y_1 + y_2)$ B, where B is a constant.

When $x = -k$ $\quad y_1 = ak^2 - bk + c$

$x = k$ $\quad y_2 = ak^2 + bk + c$

Hence, Area $= \{2\,ak^2 + 2\,c\}$ B $\qquad (2)$

and equating corresponding terms of (1) and (2) :

$$\begin{aligned} \text{B} &= h \\ \text{B}\,k^2 &= h^3/3 \\ k^2 &= h^3/3\ \text{B} = h^2/3 \\ k &= \pm\, h/\sqrt{3} \\ &= \pm\, 0{\cdot}5774\,h \end{aligned}$$

Now, B $= h = $ L/2 and, therefore, the area is found by spacing the ordinates $\pm$ 0·5774 L/2 about the centre of the base, measuring the lengths of the ordinates, and multiplying their sum by L/2, that is :

$$\text{Area ABCD} = (y_1 + y_2)\ \text{L}/2$$

When the equation of the curve is $y = ax^3 + bx^2 + cx + d$, the area is :

$$\int_{-h}^{h} y\,dx = \left[ax^4/4 + bx^3/3 + cx^2/2 + dx\right]_{-h}^{h}$$
$$= 2bh^3/3 + 2dh$$

Three ordinates are taken, one at the origin, and two spaced $\pm k$ from the origin. The area, using the lengths of these ordinates, is :

$$\left\{\begin{matrix} -ak^3 + bk^2 - ck + d \\ \qquad\qquad + d \\ +ak^3 + bk^2 + ck + d \end{matrix}\right\} \mathrm{B} = (2bk^2 + 3d)\ \mathrm{B}$$

Hence, $3\,\mathrm{B} = 2h. \qquad \mathrm{B} = 2h/3 = \mathrm{L}/3.$

$$\mathrm{B}k^2 = h^3/3$$
$$k^2 = h^3/3\ \mathrm{B} = h^2/2$$
$$k = \pm h/\sqrt{2}$$
$$= \pm 0{\cdot}7071\ h$$

and the area is $(y_1 + y_2 + y_3)\ \mathrm{L}/3.$

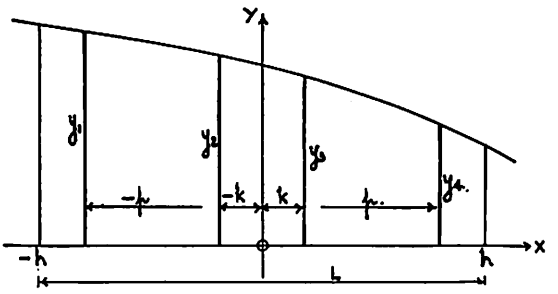

Fig. 24.

In Fig. 24, the curve is that of $y = ax^4 + bx^3 + cx^2 + dx + e$. Four ordinates y_1, y_2, y_3 and y_4 are taken, y_2 and y_3 being spaced $\mp k$, and y_1 and y_4 spaced $\mp p$ from the origin, which is again at the centre of the base.

The lengths of the ordinates are :

$$y_1 = ap^4 - bp^3 + cp^2 - dp + e$$
$$y_4 = ap^4 + bp^3 + cp^2 + dp + e$$
$$y_2 = ak^4 - bk^3 + ck^2 - dk + e$$
$$y_3 = ak^4 + bk^3 + ck^2 + dk + e$$

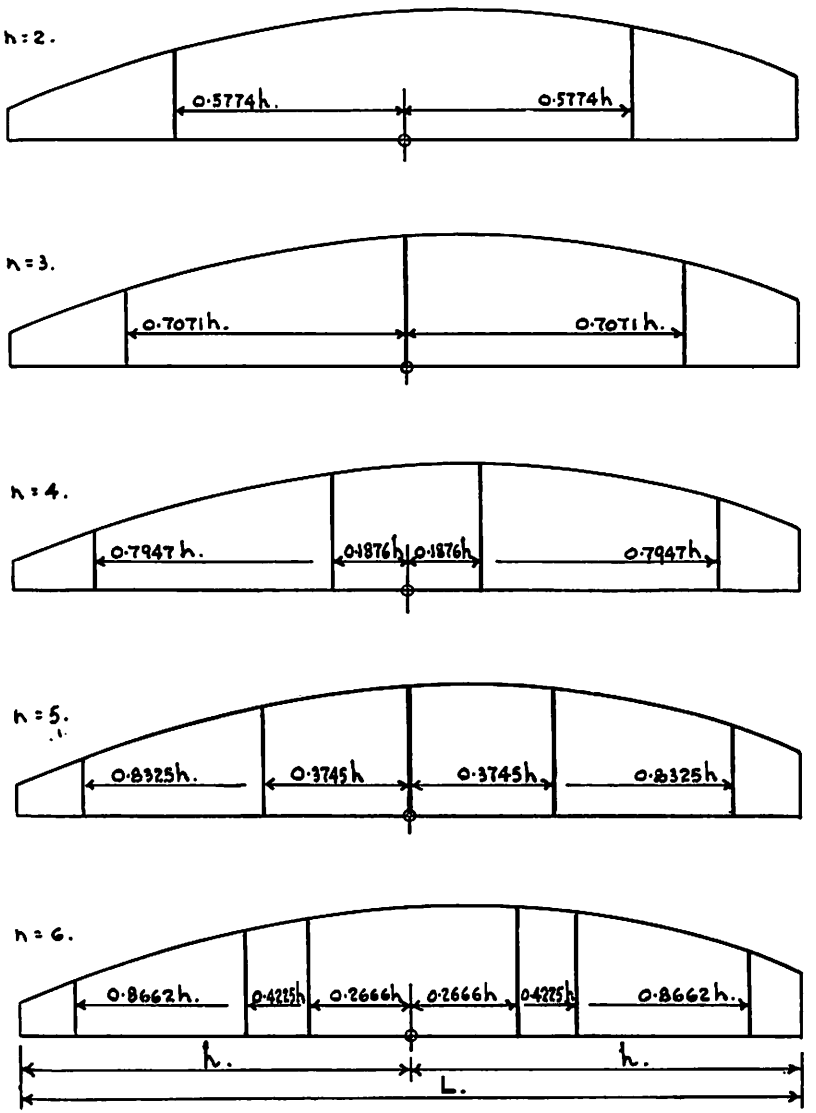
n = 2.
0·5774h.
0·5774h
n = 3.
0·7071h.
0·7071h.
n = 4.
0·7947h.
0·1876h
0·1876h
0·7947h.
n = 5.
0·8325h.
0·3745h
0·3745h
0·8325h
n = 6.
0·8662h.
0·4225h
0·2666h
0·2666h
0·4225h
0·8662h.
h.
h.
L.

Fig. 25.

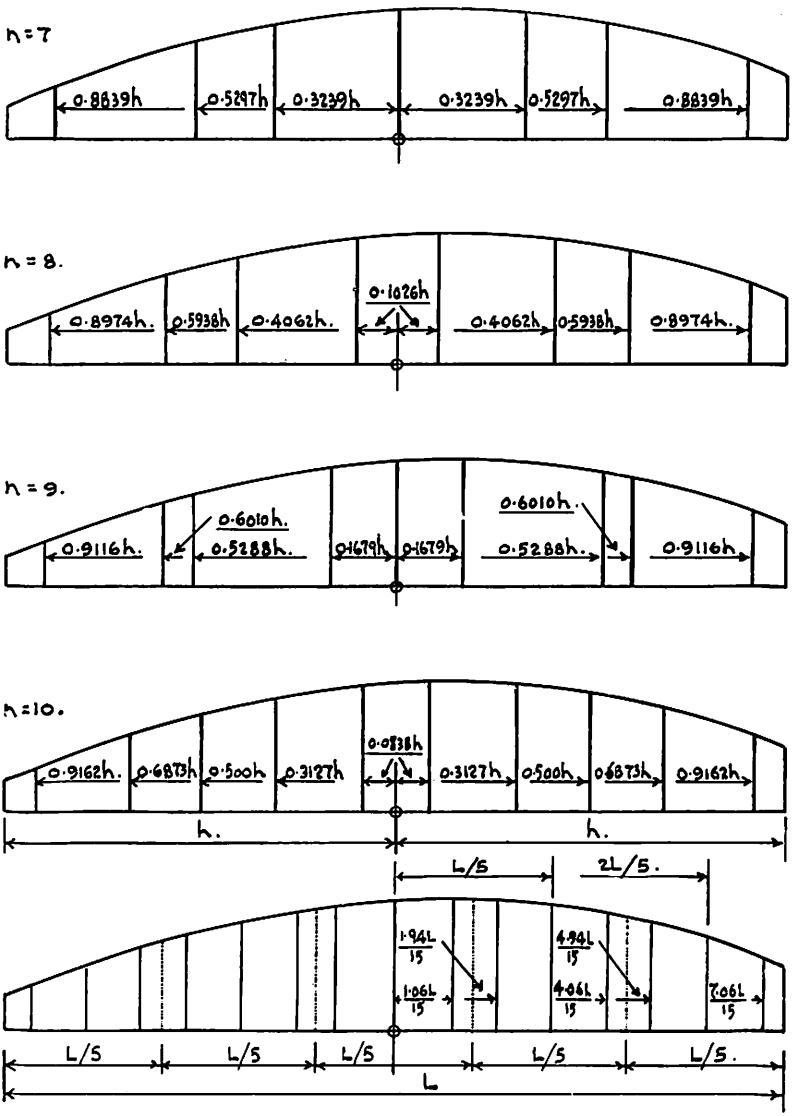
n=7
0·8839h
0·5297h
0·3239h
0·3239h
0·5297h
0·8839h
n=8.
0·8974h.
0·5938h
0·4062h.
0·1026h
0·4062h
0·5938h
0·8974h.
n=9.
0·9116h.
0·6010h.
0·5288h.
0·1679h
0·1679h
0·5288h.
0·6010h.
0·9116h
n=10.
0·9162h.
0·6873h
0·500h
0·3127h
0·0838h
0·3127h
0·500h
0·6873h
0·9162h
h.
h.
L/5
2L/5.
1·94L/15
1·06L/15
4·94L/15
4·06L/15
7·06L/15
L/5
L/5
L/5
L/5
L/5.
L

Fig. 26.

By definite integral the area is :

$$\int_{-h}^{h} y\,dx = \left[ax^5/5 + bx^4/4 + cx^3/3 + dx^2/2 + ex\right]_{-h}^{h}$$
$$= 2\,ah^5/5 + 2\,ch^3/3 + 2\,eh.$$

and by requirement the area $= (y_1 + y_2 + y_3 + y_4)$ B
$= [2a\,(k^4 + p^4) + 2c\,(k^2 + p^2) + 4e]$ B

from which $4\,B = 2\,h.$ $\quad B = 2\,h/4 = L/4$

$k^2 + p^2 = h^3/3$ B $\quad = 2\,h^2/3.$

$k^4 + p^4 = h^5/5$ B $\quad = 2\,h^4/5.$

The solution of the last two equations gives :

$k = 0{\cdot}1876\,h.$ $\quad p = 0{\cdot}7947\,h.$

The constant B is always equal to L/n, where n is the number of ordinates. These rules can be expressed $L\,(y_1+y_2+ \ldots +y_n)/n$; the part $(y_1+y_2+ \ldots .. +y_n)/n$ is the height of a mean ordinate, and the problem could have been regarded as that of spacing n ordinates so that this mean height be obtained, and hence the area.

The spacings, using up to ten ordinate, are shown in Figs. 25 and 26. It will be observed that the ordinate distances from any longitudinal point are awkward fractions of the length, and this would make the calculation of moments tedious. In one particular instance, for a special purpose, this has been obviated by dividing the base into five equal parts, as in the last diagram in Fig. 26. Three ordinates are then drawn in each of these parts. The figure now has fifteen ordinates, the spacings of which are :

2 L/5 + 0·707 L/10	= 7·06 L/15		
2 L/5	= 6·00 L/15		
2 L/5 − 0·707 L/10	= 4·94 L/15		
L/5 + 0·707 L/10	= 4·06 L/15	}	From centre of base.
L/5	= 3·00 L/15		
L/5 − 0·707 L/10	= 1·94 L/15		
Zero + 0·707 L/10	= 1·06 L/15		
Zero	= 0·0 L/15		

The numbers 7·06, 6·0, 4·94, etc., are sufficiently near to the integers 7, 6, 5, 4, 3, 2, 1 to be used as levers for moment calculations without appreciable error. It should be noticed, however, that the centre of the base is not, theoretically, a correct position for the transverse axis in the case of the middle fifth part. This can be seen by sketching the curves which would be integrated, in calculating moments, but, as the area has a long base, the error is small, and is neglected.

In the last diagram, Fig. 26, if the ordinates are as tabulated, and the length is 60 feet, then the area and centroid are calculated in the manner shown in the table overleaf :—

Length of Ordinates.	Lever.	Products for Moments.	Results.
3·5 ft.	7·0	24·5	(i) Area = 120·8 × 60/15
5·2 ,,	6·0	31·2	= 483·2 sq. ft.
6·5 ,,	5·0	32·5	
7·5 ,,	4·0	30·0	(ii) Centroid from centre of base.
8·5 ,,	3·0	25·5	
9·3 ,,	2·0	18·6	$= \frac{(221{\cdot}5 - 172{\cdot}1) \times L/15 \times L/15}{120{\cdot}8 \times L/15}$
9·8 ,,	1·0	9·8	
10·0 ,,	0·0	172·1	
10·2 ,,	1·0	10·2	$= \frac{49{\cdot}4 \times 60/15}{120{\cdot}8}$
10·1 ,,	2·0	20·2	
9·7 ,,	3·0	29·1	
9·1 ,,	4·0	36·4	= 1·64 ft. to right.
8·4 ,,	5·0	42·0	(iii) The difference between the moment products is multiplied by L/15 because the actual levers are 7 L/15, 6 L/15, etc.
7·4 ,,	6·0	44·4	
5·6 ,,	7·0	39·2	
120·8 ft.		221·5	

When longitudinal moments are not to be calculated, Tchebycheff's rules are simpler to use than any of Simpson's rules. This will be demonstrated using the solid of revolution in Fig. 27, where the volume, mass, and radius of gyration are required. Three ordinates are taken, and when spaced these give $y_1 = 6{\cdot}3$ ins., $y_2 = 7{\cdot}4$ ins., $y_3 = 8{\cdot}1$ ins., the length being 24 ins. The volume would be found by integrating the cross sectional areas longitudinally, that is,

$$\text{Vol.} = \int \pi y^2 dx.$$

The radius of gyration is required, and therefore the moment of inertia must be calculated. The solid may be considered as being made up of discs, one of which, of radius y, is shown in Fig. 27. The area of an elemental ring at distance r from the centre is

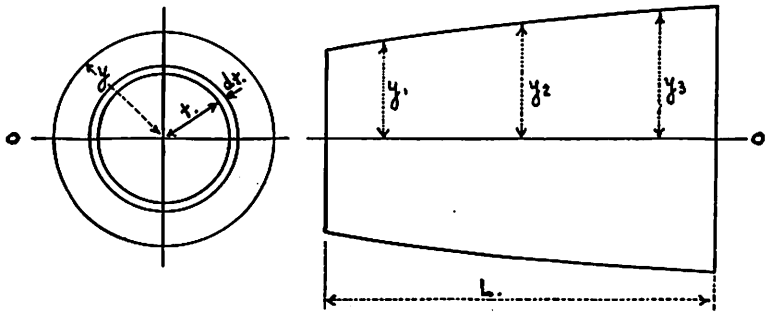

Fig. 27.

$2\pi r\,dr$, and its moment of inertia is $m \times 2\pi r\,dr \times r^2$, where m is the mass per unit area. The moment of inertia of the complete disc is

$$m\int_0^y 2\pi r^3\,dr = m\pi y^4/2$$

If y^4 were erected at each ordinate and a curve drawn, the area under this curve would be proportional to the moment of inertia of the solid. The required quantities may therefore be conveniently calculated in tabular fashion :

Ordinate	y	y^2	y^4
y_1	6·3 ins.	39·7	1575
y_2	7·4 ,,	54·8	2997
y_3	8·1 ,,	65·6	4305
	Summations	160·1	8877

(i) Volume = $(160{\cdot}1 \times \pi) \times 24/3 = 4024$ cub. ins.
(ii) Mass M = $4024\,w$. $w = wt$/cub. in.
(iii) Moment of Inertia = $w \times 8877 \times \pi/2 \times 24/3$
$= 111552\,w$ in.^{2}.

If K is the radius of gyration M K^2 = I.

(iv) $K^2 = 111{,}552/4024 = 27{\cdot}7$.
K = 5·26 ins.

(i) 160·1 is multiplied by π because the disc areas are πy^2.
(iii) 8877 is multiplied by $w\,\pi/2$ because the integral being evaluated is

$$\frac{w\ \pi}{2}\int_0^L y^4\,dx.$$

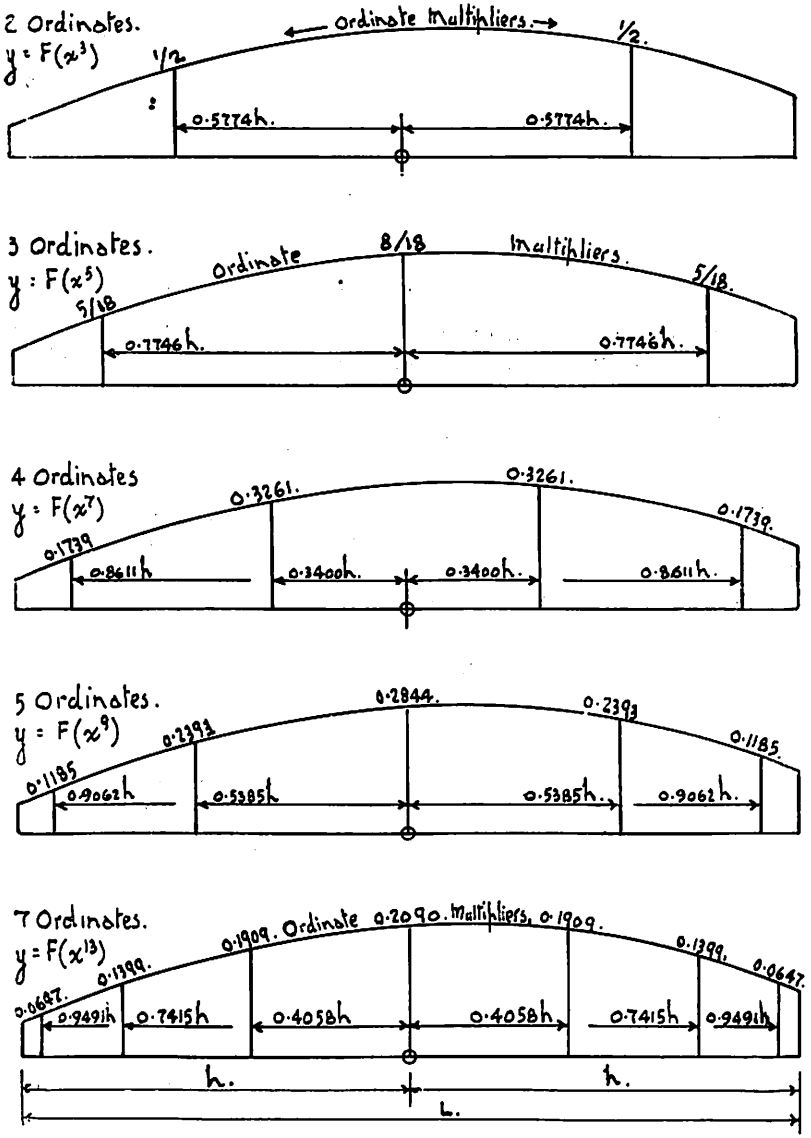
2 Ordinates.
y = F(x²)
Ordinate multipliers.
1/2
1/2
0·5774h.
0·5774h.
3 Ordinates.
y = F(x⁵)
Ordinate
8/18
multipliers.
5/18
5/18
0·7746h.
0·7746h.
4 Ordinates
y = F(x⁷)
0·3261.
0·3261.
0·1739.
0·1739
0·8611h
0·3400h.
0·3400h.
0·8611h.
5 Ordinates.
y = F(x⁹)
0·2844.
0·2393
0·2393
0·1185.
0·1185
0·9062h
0·5385h
0·5385h.
0·9062h.
7 Ordinates.
y = F(x¹³)
0·1909. Ordinate 0·2090. multipliers, 0·1909.
0·1399.
0·1399.
0·0647.
0·0647.
0·9491h
0·7415h
0·4058h
0·4058h
0·7415h
0·9491h
h.
h.
L.

Fig. 28.

Rules by Gauss.

The great German mathematician, Carl Friedrich Gauss (1777-1855) of Göttingen, took the ordinates to be spaced about the centre line of base, but with multipliers other than unity, so that there are always an even number of known conditions (for if there are m ordinates and m multipliers, the total number of conditions is $2m$). These rules, therefore, can only be derived for odd values of n, that is, for functions of the 3rd, 5th, 7th, etc., degree. The spacings and multipliers are cumbersome, and the writer does not know of any particular instance where these rules would be of advantage. The spacings and multipliers are, however, given in Fig. 28, as they may be found useful for special purposes. In the case of the 5 ordinate rule, the multipliers 0·2844, 0·2393 and 0·1185 may be replaced by the fractions 12/42, 10/42 and 5/42, so that the rule could be taken as :

$$\text{Area} = (5y_1 + 10y_2 + 12y_3 + 10y_4 + 5y_5)\ L/42,$$

and the error involved would be negligible.

Derivation of a New Rule.*

The parabola $y = ax^2 + bx + c$ requires three ordinates before the area under a section of the curve can be evaluated. Simpson took the three ordinates to be equally spaced, and deduced the 1st Rule, which has been in use for two centuries, and except for the attempts by Tchebycheff and Gauss to find simpler rules, has been accepted and used, without, so far as the author is aware, any endeavour to find better rules for special circumstances. An investigation by the author led to the deduction of a new method for performing the longitudinal integrations in ship calculations, the proof of which will now be given along with practical applications.

The three ordinates required for finding the area under a section of a 2nd order parabola need not be equally spaced.

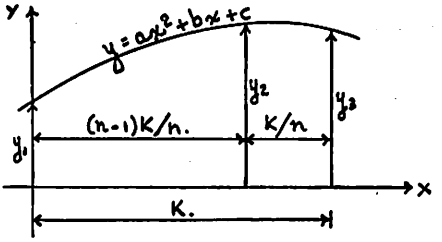

Fig. 29.

In Fig. 29 the ordinate y_2 is at a distance K/n from y_3 and therefore at a distance $(n - 1)\ K/n$ from y_1.

* The editor of "The Shipbuilder and Marine Engine-Builder" is thanked for kind permission to reprint the article entitled "A New Rule for Quadrature" by the author, which appeared in No. 448, Vol. 53 of this journal, dated June, 1946.

The area is given by :—

$$A = \int_0^K y\,dx = aK^3/3 + bK^2/2 + cK \qquad (1)$$

Assuming ordinate multipliers p, q and r, the area may be written—

$$\begin{aligned} A &= p\,y_1 + q\,y_2 + r\,y_3 \\ &= p\,c \\ &\quad + q\{aK^2\left(\frac{n-1}{n}\right)^2 + bK\left(\frac{n-1}{n}\right) + c\} \\ &\quad + r\{aK^2 + bK + c\} \\ &= c\{p + q + r\} \\ &\quad + aK^2\{q\left(\frac{n-1}{n}\right)^2 + r\} \\ &\quad + bK\{q\left(\frac{n-1}{n}\right) + r\} \end{aligned} \qquad (2)$$

Equating corresponding terms of (1) and (2)—

$$p + q + r = K$$

$$\left(\frac{n-1}{n}\right)^2 q + r = K/3$$

$$\left(\frac{n-1}{n}\right) q + r = K/2$$

Hence $$\left\{\left(\frac{n-1}{n}\right) - \left(\frac{n-1}{n}\right)^2\right\} q = K/2 - K/3 = K/6$$

$$\frac{n-1}{n}\left\{1 - \frac{n-1}{n}\right\} q = K/6$$

$$q = \frac{n^2}{6(n-1)}\,K \qquad (a)$$

$$\begin{aligned} r &= K/2 - \left(\frac{n-1}{n}\right) q \\ &= K/2 - n\,K/6 \\ &= \frac{(3-n)}{6}\,K \end{aligned} \qquad (b)$$

$$\begin{aligned} p &= K - q - r \\ &= K - \left(\frac{n^2}{n-1}\right)\frac{K}{6} - (3-n)\frac{K}{6} \end{aligned}$$

$$= \frac{6n - 6 - n^2 + n^2 - 4n + 3}{6\,(n-1)}\ \mathrm{K}$$

$$= \frac{2n - 3}{6\,(n-1)}\ \mathrm{K} \qquad \text{(c)}$$

From (b) it will be seen that r is zero when n is equal to 3, and, when this is so,

$$p = \mathrm{K}/4 \text{ and } q = 3\ \mathrm{K}/4.$$

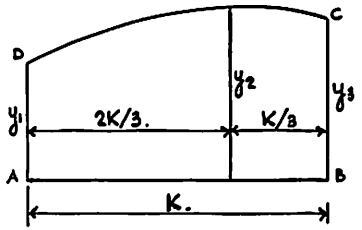

Fig. 30.

Thus the area ABCD in Fig. 30 is given by :

$$\mathrm{K}\,y_1/4 + 3\ \mathrm{K}\,y_2/4 = (y_1 + 3y_2)\ \mathrm{K}/4.$$

This two ordinate rule may be useful in certain circumstances. It is closely related to Simpson's 2nd rule, for in Fig. 31 the area ABCD is :

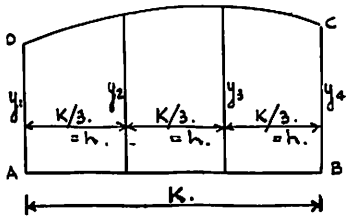

Fig. 31.

$$\begin{aligned} A &= \tfrac{1}{2}\{\ (y_1 + 3y_3)\ \mathrm{K}/4 + (3y_2 + y_4)\ \mathrm{K}/4\ \} \\ &= \{\ y_1 + 3y_2 + 3y_3 + y_4\ \}\ \mathrm{K}/8 \\ &= \{\ y_1 + 3y_2 + 3y_3 + y_4\ \}\ 3\ h/8 \end{aligned}$$

where $h = \mathrm{K}/3$.

In the equations (a), (b) and (c), let $n = 4$,

then $p = (2n - 3)\ \mathrm{K}/6\ (n-1) = 5\ \mathrm{K}/18$

$q = n^2\ \mathrm{K}/6\ (n-1) = 16\ \mathrm{K}/18$

$r = (3 - n)\ \mathrm{K}/6 = -3\ \mathrm{K}/18$

The important fact about this set of multipliers is that r has become negative. The multipliers themselves are unwieldy.

Now $(p+q+r)$ is always equal to K, and if p and r could be *numerically* equal, in which case $p = -r$, then q *must* be equal to 1·0 K and lead to a unit multiplier.

But if $p = -r$,

$$\text{then } (2n-3)/(6n-6) = -(3-n)/6$$
$$2n - 3 = n^2 - 4n + 3$$
$$n^2 - 6n + 6 = 0$$

and the solutions of this equation are

$$n = 3 \pm \sqrt{3} = 4{\cdot}732 \text{ and } 1{\cdot}268.$$

When $n = 3 + \sqrt{3}$. $\quad K/n = 0{\cdot}2113\ K$

$$p = \frac{3 + 2\sqrt{3}}{12+6\sqrt{3}}\ K$$

$$= \frac{\sqrt{3}}{6}\ K$$

$$= 0{\cdot}2887\ K$$

$$q = \frac{12+6\sqrt{3}}{2+\sqrt{3}}\ \frac{K}{6}$$

$$= 1{\cdot}0\ K$$

$$r = (3 - 3 - \sqrt{3})\ K/6$$

$$= -\frac{\sqrt{3}}{6}\ K$$

$$= -0{\cdot}2887\ K$$

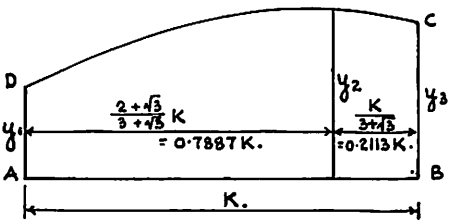

Fig. 32.

In Fig. 32 the area ABCD is therefore :

$$A = \{ K y_1 \sqrt{3}/6 + 1{\cdot}0\ K y_2 - K y_3 \sqrt{3}/6 \}$$
$$= \{ \alpha y_1 + y_2 - \alpha y_3 \}\ K, \text{ where } \alpha = \sqrt{3}/6 = 0{\cdot}2887.$$

This rule will integrate an area with an accuracy equal to that of Simpson's 1st rule, since both are derived from the second order parabola and is closely related to Tchebycheff's rule for two ordinates ; for, if (a) and (b) in Fig. 33 be identical figures—

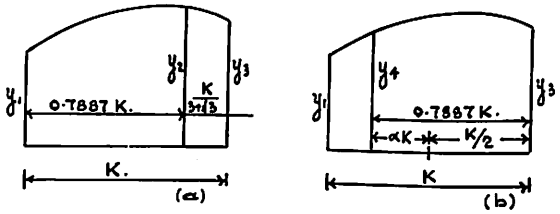

Fig. 33.

In (a), $A = \{\alpha y_1 + y_2 - \alpha y_3\}\ K$
In (b), $A = \{-\alpha y_1 + y_4 + \alpha y_3\}\ K$
whence, by addition,
$$2\ A = \{y_2 + y_4\}\ K$$
and $$A = \{y_2 + y_4\}\ K/2$$

The distance of y_2 and y_4 from the centre of base is given by :

$$d = \frac{K}{2} - \frac{K}{n} \quad (cf.\ \text{Fig. 29}).$$

and taking $n = 3 + \sqrt{3}$

$$d = K/2 - K/(3 + \sqrt{3}) = \frac{K}{2}\ \frac{1 + \sqrt{3}}{3 + \sqrt{3}}$$
$$= K/2\ \sqrt{3}$$
$$= \sqrt{3}\ K/6 = \alpha\ K$$
$$= 0{\cdot}5774\ K/2$$

which is, of course, the spacing in the Tchebycheff two-ordinate rule.

Application to Ship Calculations.

At first sight, the rule,

$$\text{Area} = \{\alpha y_1 + y_2 - \alpha y_3\}\ K$$

may not appear of much use and where areas are required using only three ordinates, Simpson's 1st rule would undoubtedly be better ; the value of α (viz., 0·2887) makes the rule cumbersome for three ordinates. The method, however, was devised for use in integrating long figures such as occur in ship calculations.

Fig. 34 represents the forward end of the water-plane of a ship. The upper part (a) shows the usual equi-distant stations for the application of Simpson's 1st rule. In the lower part (b), the same number of ordinates appear, but they are erected to suit the new rule, according to which the area is given by—

$$A = \{\alpha y_1 + y_2 - \alpha y_3\}\ K + \{\alpha y_3 + y_4 - \alpha y_5\}\ K + \ldots.$$
$$+ \ldots. + \{\alpha y_9 + y_{10} - \alpha y_{11}\}\ K$$
$$= \{\alpha y_1 + y_2 + y_4 + y_6 + y_8 + y_{10} - \alpha y_{11}\}\ K$$

Now $(p+q+r)$ is always equal to K, and if p and r could be *numerically* equal, in which case $p = -r$, then q *must* be equal to 1·0 K and lead to a unit multiplier.

But if $p = -r$,

then $(2n-3)/(6n-6) = -(3-n)/6$

$2n - 3 = n^2 - 4n + 3$

$n^2 - 6n + 6 = 0$

and the solutions of this equation are

$$n = 3 \pm \sqrt{3} = 4{\cdot}732 \text{ and } 1{\cdot}268.$$

When $n = 3 + \sqrt{3}$. $\quad K/n = 0{\cdot}2113\ K$

$$p = \frac{3 + 2\sqrt{3}}{12 + 6\sqrt{3}}\ K$$

$$= \frac{\sqrt{3}}{6}\ K$$

$$= 0{\cdot}2887\ K$$

$$q = \frac{12 + 6\sqrt{3}}{2 + \sqrt{3}}\ \frac{K}{6}$$

$$= 1{\cdot}0\ K$$

$$r = (3 - 3 - \sqrt{3})\ K/6$$

$$= -\frac{\sqrt{3}}{6}\ K$$

$$= -0{\cdot}2887\ K$$

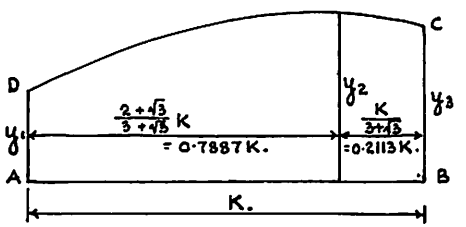

Fig. 32.

In Fig. 32 the area ABCD is therefore :

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$$= \{ \alpha y_1 + y_2 - \alpha y_3 \}\ K, \text{ where } \alpha = \sqrt{3}/6 = 0{\cdot}2887.$$

This rule will integrate an area with an accuracy equal to that of Simpson's 1st rule, since both are derived from the second order parabola and is closely related to Tchebycheff's rule for two ordinates ; for, if (a) and (b) in Fig. 33 be identical figures—

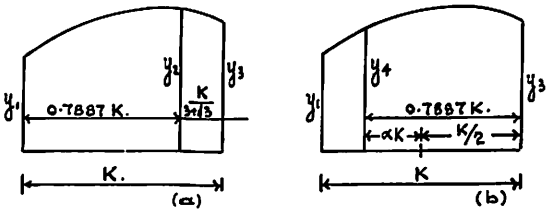

Fig. 33.

In (a), $A = \{\alpha y_1 + y_2 - \alpha y_3\}\ K$
In (b), $A = \{-\alpha y_1 + y_4 + \alpha y_3\}\ K$
whence, by addition,

$$2\ A = \{y_2 + y_4\}\ K$$
and $$A = \{y_2 + y_4\}\ K/2$$

The distance of y_2 and y_4 from the centre of base is given by :

$$d = \frac{K}{2} - \frac{K}{n} \text{ (cf. Fig. 29).}$$

and taking $n = 3 + \sqrt{3}$

$$d = K/2 - K/(3 + \sqrt{3}) = \frac{K}{2}\ \frac{1 + \sqrt{3}}{3 + \sqrt{3}}$$
$$= K/2\ \sqrt{3}$$
$$= \sqrt{3}\ K/6 = \alpha\ K$$
$$= 0{\cdot}5774\ K/2$$

which is, of course, the spacing in the Tchebycheff two-ordinate rule.

Application to Ship Calculations.

At first sight, the rule,

$$\text{Area} = \{\alpha y_1 + y_2 - \alpha y_3\}\ K$$

may not appear of much use and where areas are required using only three ordinates, Simpson's 1st rule would undoubtedly be better ; the value of α (viz., 0·2887) makes the rule cumbersome for three ordinates. The method, however, was devised for use in integrating long figures such as occur in ship calculations.

Fig. 34 represents the forward end of the water-plane of a ship. The upper part (a) shows the usual equi-distant stations for the application of Simpson's 1st rule. In the lower part (b), the same number of ordinates appear, but they are erected to suit the new rule, according to which the area is given by—

$$A = \{\alpha y_1 + y_2 - \alpha y_3\}\ K + \{\alpha y_3 + y_4 - \alpha y_5\}\ K + \ldots.$$
$$+ \ldots. + \{\alpha y_9 + y_{10} - \alpha y_{11}\}\ K$$
$$= \{\alpha y_1 + y_2 + y_4 + y_6 + y_8 + y_{10} - \alpha y_{11}\}\ K$$

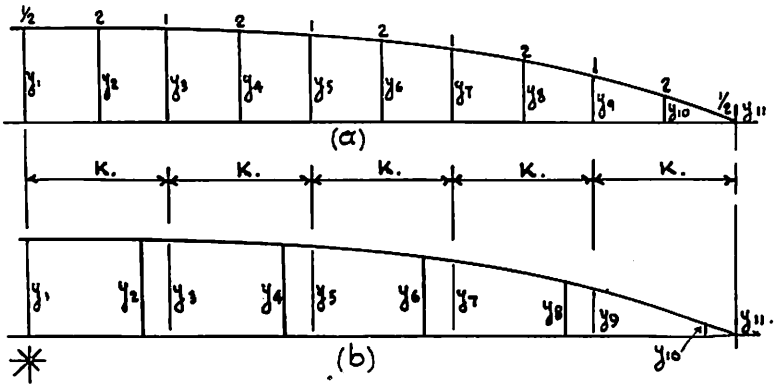

Fig. 34.

It will be seen that the terms containing y_3, y_5, y_7 and y_9 are eliminated, which is the object of using the rule in this manner. In general, y_{11} will be zero, so that the expression for the area finally takes the form :

$$A = \{ \alpha y_1 + y_2 + y_4 + y_6 + y_8 + y_{10} \} K.$$

The multiplier α associated with y_1 is eliminated as follows. In Fig. 35,

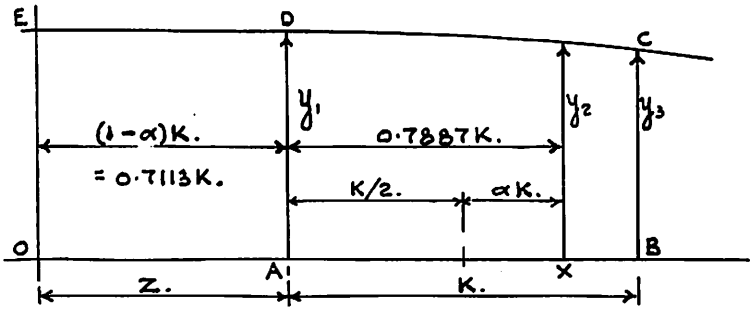

Fig. 35.

$$\text{Area ABCD} = \{ \alpha y_1 + y_2 - \alpha y_3 \} K$$

$$\text{Take OA} = Z = (1 - \alpha) K = 0{\cdot}7113 K$$

Then, since AD will in general be equal to OE (or nearly so), both being at about amidships,

$$\text{Area OADE} = (1 - \alpha) K y_1$$

$$\text{Area OBCE} = \{ (1 - \alpha) y_1 + \alpha y_1 + y_2 - \alpha y_3 \} K$$

$$= \{ y_1 + y_2 - \alpha y_3 \} K$$

and y_1 now has a unit multiplier.

In ship calculations, 1st and 2nd moments of area are required about the midship ordinate OE. The distance OX in Fig. 35 is, therefore, of supreme importance.

$$\text{Now OA} = (1 - \alpha)\ \text{K} = (1 - \sqrt{3}/6)\ \text{K}$$

$$= \left(\frac{6 - \sqrt{3}}{6}\right) \text{K}$$

$$\text{and AX} = \text{K} - \text{K}/n = \frac{n - 1}{n}\ \text{K}$$

$$= \frac{2 + \sqrt{3}}{3 + \sqrt{3}}\ \text{K}$$

whence,

$$\text{OX} = \text{OA} + \text{AX} = \left\{\frac{6 - \sqrt{3}}{6} + \frac{2 + \sqrt{3}}{3 + \sqrt{3}}\right\} \text{K}$$

$$= \frac{27 + 9\sqrt{3}}{18 + 6\sqrt{3}}\ \text{K}$$

$$= 3\ \text{K}/2$$

If the ordinate y_2 could be placed at a distance 2 K from the midship position it would be an advantage and simplify the moment calculations. This simplification may be secured by taking an ordinate y_0 at midships with a distance K/2 reserved on either side of it before setting out the ordinates at the spacing for the new rule. By this means the first interval is (0·5 K + 0·7113 K), *i.e.*, 1·2113 K and the second 0·7887 K, followed by *any* required number of spaces each of length K.

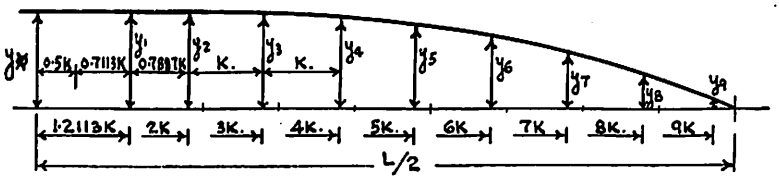

Fig. 36.

Fig. 36 shows the spacing when there are ten ordinates forward of the midship ordinate, the tenth being zero.

$$\text{Here } 0{\cdot}5\ \text{K} + 0{\cdot}7113\ \text{K} + 8\ \text{K} = \text{L}/2$$

$$9{\cdot}2113\ \text{K} = \text{L}/2$$

$$\text{K} = \text{L}/18{\cdot}4226$$

$$\text{and A (Forward)} = \{y_0/2 + y_1 + y_2 + \ldots + y_9\}\ \text{K} \qquad (1)$$

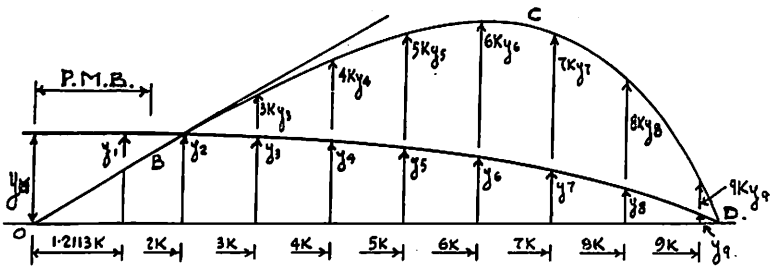

Fig. 37.

The First Moment.

The curve OBCD in Fig. 37 represents the forward curve which would have to be integrated in evaluating the first moment about the midship ordinate. There is a similar curve aft and the moment preponderance forward or aft determines the position of the centre of flotation. In a ship having parallel middle body (P.M.B.) the part OB is a straight line, tangential to the moment curve. The area under OBCD represents the moment of area forward of midships about the midship ordinate.

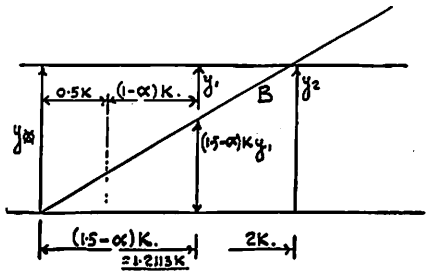

Fig. 38.

Fig. 38 is an enlargement of Fig. 37 in the region of amidships, and from these two figures, the 1st Moment about the midship ordinate is given by :—

$$M_1 = (1.5 - \alpha)\, K\, (1{\cdot}5 - \alpha)\, K y_1/2 + K\,[(1{\cdot}5 - \alpha)\, K y_1 \alpha + 2\,K y_2 + 3\,K y_3 + \ldots + 8\,K y_8 + 9\,K y_9]$$

$$= K^2 \left[\frac{1{\cdot}5^2 - \alpha^2}{2} y_1 + 2y_2 + 3y_3 + \ldots + 8y_8 + 9y_9 \right]$$

$$= K^2\,[1{\cdot}083\, y_1 + 2y_2 + 3y_3 + \ldots + 8y_8 + 9y_9]$$

which, without loss of accuracy, may be taken as,

$$M_1 = K^2\,[1{\cdot}0\, y_1 + 2y_2 + 3y_3 + \ldots + 8y_8 + 9y_9] \qquad (2)$$

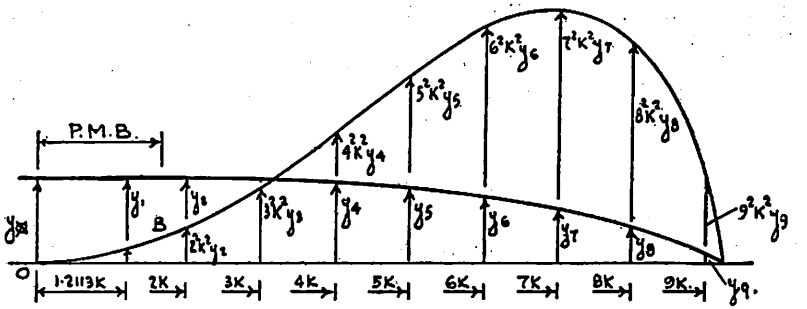

Fig. 39.

The Second Moment.*

Fig. 39 has been drawn to show the forward curve to be integrated in the calculation of the 2nd moment. There would be a corresponding curve aft and the integrations of the two curves would be added to obtain the 2nd moment.

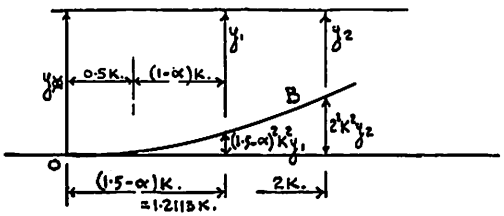

Fig. 40.

Fig. 40 is an enlargement of Fig. 39 in the neighbourhood of amidships. In the range of parallel middle body the curve OB in Fig. 40 would be a 2nd order parabola and the 2nd moment of the forward portion of the water-plane about the midship ordinate is given by :—

$$M_2 = (1{\cdot}5 - \alpha)\ K\ (1{\cdot}5 - \alpha)^2\ K^2 y_1/3 + K\ [(1{\cdot}5 - \alpha)^2\ K^2 y_1 \alpha + 2^2K^2y_2 + 3^2K^2y_3 + \ldots . \ldots . + 8^2K^2y_8 + 9^2K^2y_9]$$

$$= K^3\ [1{\cdot}016\, y_1 + 2^2y_2 + 3^2y_3 + \ldots . + 8^2y_8 + 9^2y_9]$$

and this may be taken as :—

$$M_2 = K^3\ [1^2y_1 + 2^2y_2 + 3^2y_3 + \ldots + 8^2y_8 + 9^2y_9] \qquad (3)$$

An examination of figures 37, 38, 39, 40 and equations (1), (2) and (3) will show that the ordinate y_1 does not require to be maintained in the position 1·2113 K from y_0 and that no serious sacrifice of accuracy is involved if it is taken at a distance K from y_0. The

* More usually called the Moment of Inertia.

ordinates are then equally spaced so that the spacing is better than in any of the Tchebycheff rules.

It is suggested that eight "K Spaces" be adopted, forward and aft of the midship ordinate, as the standard in applying the method, irrespective of the size of the vessel. In this case,

$$8\ K + 0{\cdot}2113\ K = L/2$$
$$K = L/16{\cdot}4226$$

The ordinates would be spaced as shown at (d) in Fig. 41, which illustrates the arrangement of ordinates to suit various methods of integration, viz.,

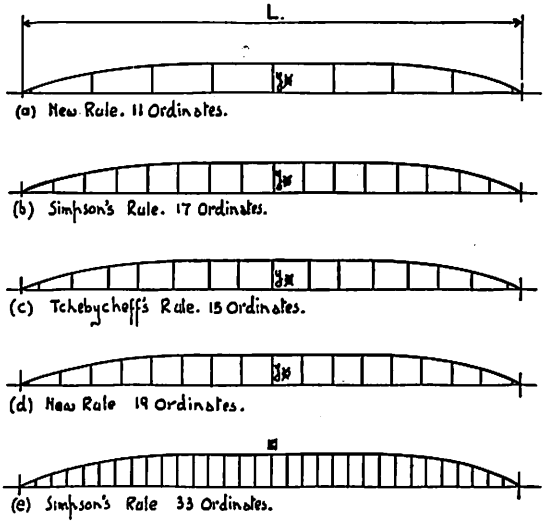
L.
(a) New Rule. 11 Ordinates.
(b) Simpson's Rule. 17 Ordinates.
(c) Tchebycheff's Rule. 15 Ordinates.
(d) New Rule 19 Ordinates.
(e) Simpson's Rule 33 Ordinates.

Fig. 41.

(a)	New Rule, ...	11	Ordinates.
(b)	Simpson's Rule, ...	17	,,
(c)	Tchebycheff's Rule, ...	15	,,
(d)	New Rule, ...	19	,,
(e)	Simpson's 1st Rule, ...	33	,,

Standard calculation sheets are sometimes used in conjunction with Tchebycheff spacings using 15 ordinates (c) which is a modification of the three-ordinate rule and these sheets could be used without alteration if the new method is adopted, also using 15 (real) ordinates. The writer, however, advocates an equal number of K spaces.

The ordinate lengths were scaled from the outline curve and the calculation of area, position of centre of flotation and the longitudinal 2nd moment performed in the tabular manner usual in these computations as follows—

TABLE I.—Calculations for Fig. 41.

(*a*) New Rule. 4·2113 K = 100. K = 100/4.2113 ft.

Number of Ordinate (x)	F (x) = [y] Ft.	xF (x) = [xy]	x^2 F (x) = [x^2y]
4	2·21	8·84	35·36
3	8·80	26·40	79·20
Aft. 2	11·49	22·98	45·96
1	12·40	12·40	12·40
0	12·50	70·62	0·0
1	12·50	12·50	12·50
Frd. 2	12·00	24·00	48·00
3	9·79	29·37	88·11
4	2·79	11·16	44·64
	Σ 84·48	77·03	366·17

(The No. 5 Ordinates are of zero length).

Log 100 = 2·00000
„ 4·2113 = 0·62442
„ K = 1·37558
„ 84·48 = 1·92675
„ Area = 3·30233
Area = 2006·0 sq. ft.

$M_2 = 366{\cdot}17\ K^3$

$M_2 = 4902600$ ft.^{4}

$G = \{77{\cdot}03 - 70{\cdot}62\}\ K/84{\cdot}48.$
Log 6·41 = 0·80686
„ K = 1·37558
„ 6·41 K = 2·18244
„ 84·48 = 1·92675
„ G = 0·25569
G = 1·80 ft.

Log K^3 = 4·12674
„ 366·17 = 2·56368
Log M_2 = 6·69042

TABLE II.—Calculations for Fig. 41.

(*b*) Simpson's 1st Rule. Common Interval = 100/8 ft.

No. of Ord.	Length of Ord. Ft.	S M	F (A)	F (1st Mt.)	F (2nd Mt.)
8	0·00	½	0·00	0·0	0·0
7	5·00	2	10·00	70·0	490·0
6	8·10	1	8·10	48·6	291·6
5	10·10	2	20·20	101·0	505·0
Aft. 4	11·30	1	11·30	45·2	180·8
3	12·05	2	24·10	72·3	216·9
2	12·40	1	12·40	24·8	49·6
1	12·50	2	25·00	25·0	25·0
0	12·50	1	12·50	386·9	0·0
1	12·50	2	25·00	25·0	25·0
Frd. 2	12·50	1	12·50	25·0	50·0
3	12·35	2	24·70	74·1	222·3
4	11·90	1	11·90	47·6	190·4
5	11·00	2	22·00	110·0	550·0
6	9·15	1	9·15	54·9	329·4
7	5·95	2	11·90	83·3	583·1
8	0·0	½	0·0	0·0	0·0
			240·75	419·9	3709·1

$$\text{Area} = \frac{2}{3}\ \frac{100}{8}\ 240{\cdot}75$$
$$= 2006{\cdot}2 \text{ sq. ft.}$$

$$G = (419{\cdot}9 - 386{\cdot}9) \times 12{\cdot}5/240{\cdot}75$$
$$= 1{\cdot}71 \text{ ft.}$$

$$M_2 = \frac{2}{3}\ 3709{\cdot}1 \times 12{\cdot}5^3$$
$$= 2472{\cdot}7 \times 12{\cdot}5^3$$
$$= 4829400 \text{ ft.}^4$$

Log $12{\cdot}5^3$ = 3·29073
,, 2472·7 = 3·39317
,, M_2 = 6·68390

TABLE III.—Calculation for Fig. 41.

(*c*) Tchebycheff's Rule. 15 Ordinates.

No. of Ordinates.	Length of Ord. Ft.	F (1st Mt.).	F (2nd Mt.).
Aft. 7	2·60	18·20	127·4
6	7·00	42·00	252·0
5	9·61	48·05	240·25
4	10·95	43·80	175·2
3	11·91	35·73	107·19
2	12·40	24·80	49·6
1	12·50	12·50	12·5
0	12·50	225·08	0·0
Frd. 1	12·50	12·50	12·5
2	12·50	25·00	50·0
3	12·30	36·90	110·7
4	11·65	46·60	186·4
5	10·59	52·95	264·75
6	8·10	48·60	291·6
7	3·25	22·75	159·25
	150·36	245·30	2039·34

Log 200 = 2·30103
„ 15 = 1·17609
„ 200/15 = 1·12494
Log 150·36 = 2·17713
„ Area = 3·30207

Area = 2004·8 sq. ft.

M_2 = 2039·34 (200/15)³
= 4833900 ft.⁴ units

G = (245·30 − 225·08) 200/15 × 150·36
= 1·79 ft.

Log 200/15 = 1·12494
„ 20·22 = 1·30578
„ 20·22 × 200/15 = 2·43072
Log 150·36 = 2·17713
„ G = 0·25359

Log (200/15)³ = 3·37482
„ 2039·3 = 3·30948
„ M_2 = 6·68430

TABLE IV.—Calculation for Figure 41.

(*d*) New Rule. 8·2113 K = 100. K = 100/8·2113 ft.

No. of Ordinate.	Length of Ord. Ft.	F (1st Mt.).	F (2nd Mt.).
Aft. 8	1·15	9·20	73·6
7	5·60	39·2	274·4
6	8·40	50·4	302·4
5	10·36	51·8	259·0
4	11·40	45·6	182·4
3	12·10	36·3	108·9
2	12·41	24·82	49·64
1	12·50	12·50	12·5
0	12·50	269·82	0·0
Frd. 1	12·50	12·5	12·5
2	12·50	25·0	50·0
3	12·40	37·2	111·6
4	11·98	47·92	191·68
5	11·11	55·55	277·75
6	9·49	56·94	341·64
7	6·63	46·41	324·87
8	1·50	12·0	96·00
	164·53	293·52	2668·88

(The No. 9 Ordinates are of zero length).

Log 100 = 2·00000
„ 8·2113 = ·91441
„ K = 1·08559
„ 164·53 = 2·21625
„ Area = 3·30184
Area = 2003·7 sq. ft.

$M_2 = 2668{\cdot}9\ K^3$
$= 4820600$ ft.^{4} units.

$G = (293{\cdot}52 - 269{\cdot}82)\ K/164{\cdot}53$
Log 23·7 = 1·37475
„ K = 1·08559
„ 23·7 K = 2·46034
„ 164·53 = 2·21625
„ G = 0·24409
G = 1·75 ft.

Log K^3 = 3·25677
„ 2668·9 = 3·42633
„ M_2 = 6·68310

TABLE V.—Calculation for Fig. 41.

(*e*) Simpson's 1st Rule. Common Interval = 100/16 ft.

No. of Ordinate.		Length of Ordinate Ft.	S.M.	F (A)	F (1st Mt.)	F (2nd Mt.)
Aft.	16	0·0	½	0·0	0·0	0·0
	15	2·80	2	5·6	84·0	1260·0
	14	5·0	1	5·0	70·0	980·0
	13	6·68	2	13·36	173·68	2257·84
	12	8·10	1	8·1	97·2	1166·4
	11	9·15	2	18·3	201·3	2214·3
	10	10·10	1	10·1	101·0	1010·0
	9	10·71	2	21·42	192·78	1735·02
	8	11·30	1	11·3	90·4	723·2
	7	11·70	2	23·4	163·8	1146·6
	6	12·05	1	12·05	72·3	433·8
	5	12·25	2	24·5	122·5	612·5
	4	12·40	1	12·4	49·6	198·4
	3	12·45	2	24·9	74·7	224·1
	2	12·50	1	12·5	25·0	50·0
	1	12·50	2	25·0	25·0	25·0
	0	12·50	1	12·5	1543·26	0·0
Frd.	1	12·50	2	25·0	25·0	25·0
	2	12·50	1	12·5	25·0	50·0
	3	12·50	2	25·0	75·0	225·0
	4	12·50	1	12·5	50·0	200·0
	5	12·46	2	24·92	124·6	623·0
	6	12·35	1	12·35	74·1	444·6
	7	12·18	2	24·36	170·52	1193·64
	8	11·90	1	11·9	95·2	761·6
	9	11·50	2	23·0	207·0	1863·0
	10	11·00	1	11·0	110·0	1100·0
	11	10·20	2	20·4	224·4	2468·4
	12	9·15	1	9·15	109·8	1317·6
	13	7·75	2	15·5	201·5	2619·5
	14	5·95	1	5·95	83·3	1166·2
	15	3·50	2	7·0	105·0	1575·0
	16	0·00	½	0·0	0·0	0·0
				480·96	1680·42	29669·7

$$\text{Area} = \frac{2}{3}\ \frac{100}{16}\ 480{\cdot}96 \qquad G = (1680{\cdot}42 - 1543{\cdot}26)\ 6{\cdot}25/480{\cdot}96$$

$$= 2004{\cdot}0 \text{ sq. ft.} \qquad = 1{\cdot}78 \text{ ft.}$$

$$M_2 = \frac{2}{3}\ 29669{\cdot}7 \times 6{\cdot}25^3 \qquad \text{Log } 6{\cdot}25^3 = 2{\cdot}38764$$

$$= 19780 \times 6{\cdot}25^3 \qquad ,, \quad 19780 = 4{\cdot}29623$$

$$= 4829100 \text{ ft.}^4 \text{ units.} \qquad ,, \quad M_x = 6{\cdot}68387$$

TABLE VI.—Summary of Results.

Method.	No. of Ordinates.	Area.	G.	M_2
		Sq. ft.	Ft.	Ft.^{4}
(*a*) New Rule,	11	2006·0	1·80	4902600
(*b*) Simpson's 1st,	17	2006·2	1·71	4829400
(*c*) Tchebycheff,	15	2004·8	1·79	4833900
(*d*) New Rule,	19	2003·7	1·75	4820600
(*e*) Simpson's 1st,	33	2004·0	1·78	4829100

The deductions $A \times G^2$ where A is the area and G the distance of the centroid from the midship ordinate, have not been made in the second moment calculations being irrelevant to the present discussion.

It will be seen that there is much more arithmetical work involved in using Simpson's rule than in either Tchebycheff's or the new rule. Tchebycheff's rule, however, requires ordinates at awkward spacing, while those required for the new rule, being equally spaced, are more easily set out. The summary in Table VI. appears to confirm that, as against the new rule, Simpson's 1st rule requires twice the number of ordinates for equal accuracy.

The new rule, with eight K spaces, Fig. 41 (a) should be useful in the early stages of design, as it is comparatively accurate for a small number of ordinates. Care is required in its application, however, and, when the last ordinate is not zero, so that it must be multiplied by $-\alpha$ the integrations may be arranged as indicated in Table VII., which refers to Fig. 42. The actual centroid of the area ABCD and the 2nd moments about the YY and XX axis are required. The example is chosen for its illustrative rather than its practical use. The ordinates y_1 and y_2 are equal, and therefore the rule previously deduced for equal spacing is employed.

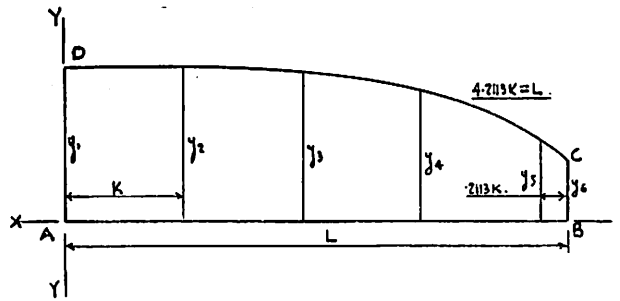

Fig. 42.

TABLE VII.—Calculation for Fig. 42.

(*a*) Longitudinal. Length = 40 ft. K = 40/4·2113 ft.

Length of Ordinate.	M.	F (A) Ft.	Lever.	F (1st Mt.)	Lever.	F (2nd Mt.)
y_1 12·6	½	6·3	0	0·0	0	0·0
y_2 12·6	1	12·6	1	12·6	1	12·6
y_3 12·0	1	12·0	2	24·0	2	48·0
y_4 10·4	1	10·4	3	31·2	3	93·6
y_5 6·4	1	6·4	4	25·6	4	102·4
		47·7		93·4		256·6
y_6 4·8	− 0·289	− 1·4	4·2113	− 5·9	4·2113	− 24·8
		46·3		87·5		231·8

Area = 46·3 × 40/4·2113
= 439·8 sq. ft.

M_1 = 87·5 × (40/4·2113)²
= 7896 ft.³ units.

G = 7896/439·8
= 18·0 ft.

M_2 = 231·8 × (40/4·2113)³
= 198,700 ft.⁴ units.

(*b*) Transverse. *

* The integrals being evaluated are :—

$$M_1 = \int_0^L \tfrac{1}{2} y^2 \, dx \text{ and } M_2 = \int_0^L \tfrac{1}{3} y^3 \, dx$$

Length of Ordinate Ft.	Ord.^{2}	M.	F (y^2)	Ord.^{3}	M.	F (y^3)
y_1 12·6	158·8	½	79·4	2000	½	1000
y_2 12·6	158·8	1	158·8	2000	1	2000
y_3 12·0	144·0	1	144·0	1728	1	1728
y_4 10·4	108·2	1	108·2	1125	1	1125
y_5 6·4	41·0	1	41·0	262	1	262
			531·4			6115
y_6 4·8	23·0	− 0·289	− 6·6	111	− 0·289	− 32
			524·8			6083

$M_1 = 524{\cdot}8 \times 40/4{\cdot}2113 \times 2.$
$= 2493$ ft.^{3} units.

$G = 2493/439{\cdot}8$
$= 5{\cdot}66$ ft.

$M_2 = 6083 \times 40/4{\cdot}2113 \times 3$
$= 19{,}270$ ft.^{4} units.

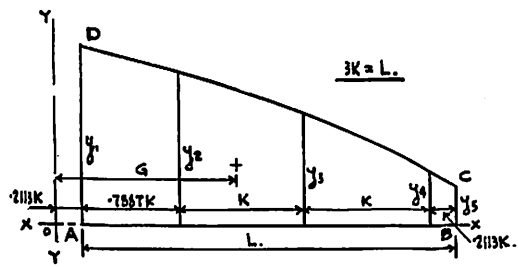

Fig. 43.

Where the ordinates y_1 and y_2 are unequal as at ABCD in Fig. 43, the length is divided to suit the rule—

$$A = (\alpha y_1 + y_2 + \ldots\ldots + y_{n-1} - \alpha y_n)\ K.$$

It has been stated that the spacing of the ordinates is awkward for the calculation of moments, but this difficulty can be overcome by taking the YY axis outside the figure at a distance $(K/2 - \alpha K) = 0{\cdot}2113\ K$ from y_1. The levers of the ordinates, except the first and last, are then multiples of the K value, and the integrations are easily performed in the usual tabular manner. The details are given in Table VIII.

TABLE VIII.—Calculation for Fig. 43.

Length = 30 ft. K = 30/3 = 10 ft.

Length of Ordinate Ft.	M.	F (A)	Lever.	F (xy)	M.	F (M_1)
y_1 14·3	0·289	4·1	0·2113	3·0	0·289	0·9
y_2 12·4	1·0	12·4	1·0	12·4	1·0	12·4
y_3 9·2	1·0	9·2	2·0	18·4	1·0	18·4
y_4 4·4	1·0	4·4	3·0	13·2	1·0	13·2
		30·1				44·9
y_5 3·1	− 0·289	− 0·9	3·2113	10·0	− 0·289	− 2·9
		29·2				42·0

Area = 29·2 × 10 = 292 sq. ft.

$M_1 = 42 \times 10^2 = 4200$ ft.³ units.

G = 4200/292 = 14·4 ft. from YY.

Graphic Integration.

The process of graphic integration is useful when approximate integral curves are required without actual calculation.

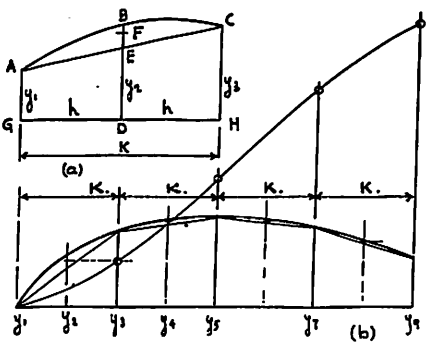

Fig. 44.

In Fig. 44 (a),

$$\text{Area AECB} = h y_1/3 + 4 h y_2/3 + h y_3/3 - h y_1 - h y_3$$
$$= 4 h y_2/3 - 2 h y_1/3 - 2 h y_3/3$$
$$= \{y_2 - (y_1/2 + y_3/2)\}\ 4 h/3$$
$$= \{DB - DE\}\ 2K/3$$
$$= 2K \cdot EB/3$$

Hence, if the point F be spotted off such that EF = 2 EB/3, then DF is the mean height of the figure, and DF × K is the area of GHCBA.

In Fig. 44 (b) the base is divided into any suitable number of equally spaced ordinates. The tops of the ordinates are joined and the lines are such as AC in (a). The points F at two-thirds of EB are located. Commencing at the first space, all such heights as DF are measured successively on a strip of paper. These automatic additions are then spotted off on the ordinates y_3, y_5, y_7, etc. The curve drawn through these points is the 1st integral curve to some scale, and the area may be obtained by scaling the height of the integral curve (using the same scale to which the fundamental curve is drawn) and multiplying by the length K.

A better method is to use Tchebycheff's rule for two ordinates. The base, as before, is divided into a suitable number of equal spaces. Each of these spaces then has the two ordinates drawn in at the correct spacing, and if the tops of these ordinates are joined the line intersects the vertical through the centre of the (part) base at the mean height of the relevant part of the figure. The method is very well illustrated by considering an example where a certain part of a mechanism had to have its maximum velocity v at 4T/5 on the forward movement, T being the time taken during the forward movement. The graph of the velocity is to be as shown in Fig. 45 (a). The space-time curve is required, and the maximum velocity deduced if T = $1\frac{1}{2}$ seconds and the travel 3 inches.

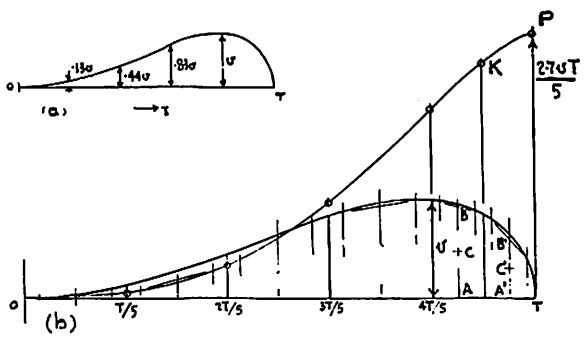

Fig. 45.

The velocity time curve is drawn to a greater scale (b) and the area under this curve represents the travel or distance covered. The integral curve commences at 0, and any ordinate of this curve gives the distance covered in the appropriate time up to that ordinate. In each space representing T/5, set up the two Tchebycheff ordinates and also the centre line of the space. The line

joining the tops of the ordinates cuts the centre line of the space at the mean height of the space. On a strip of paper add these mean heights successively, and use as ordinates for the integral curve. The last space has been divided into two parts, each T/10. Therefore, the areas are AB × L/10 = AB/2 × L/5. Bisect AB in C and use AC when taking off the heights on the strip of paper to obtain the points K and P on the space-time curve.

The height TP is measured on the same scale as the fundamental curve and is 2·7 v. Hence 2·7 v T/5 is the travel and

$$2{\cdot}7\ v\ \mathrm{T}/5 = 3 \text{ inches.}$$

$$v = \frac{3 \times 5}{2{\cdot}7\ \mathrm{T}} = \frac{3 \times 5}{2{\cdot}7 \times 1{\cdot}5}$$

$$= 3{\cdot}7 \text{ inches/sec.}$$

Polar Co-ordinates.

The Cartesian co-ordinates of a point are the abscissa x and the ordinate y, and this system has been used for specifying the various functions dealt with.

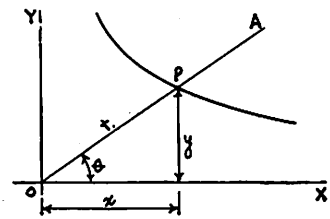

Fig. 46.

In Fig. 46 the point P has Cartesian co-ordinates (x, y). If the origin is now regarded as a pole, and that OP is of length r, it is evident that the point P could be plotted by drawing the angle θ, and along the line OA, setting off the length r.

The *polar co-ordinates* of P are (r, θ) and

$$r = \sqrt{(x^2 + y^2)}$$

$$\theta = \tan^{-1}(y/x).$$

Any function may therefore be transformed from Cartesian to polar co-ordinates by the substitution of $x = r \cos \theta$ and $y = r \sin \theta$ in its equation and vice-versa. The length r is known as a radius vector, and in some cases, the use of polar co-ordinates leads to much easier graphical representation of functions.

Radian Measure of Angles.

The angle subtended at the centre of a circle by an arc equal in length to the radius is the natural unit for angles and is called a

radian. The symbol π is given to the radian measure of two right angles, *i.e.*,

$$\pi \text{ radians} = 180°$$
$$1 \text{ radian} = 180°/\pi$$
$$= 57° \ 17' \ 45'' \text{ nearly.}$$

All problems using radius vectors must be dealt with, using radian measure.

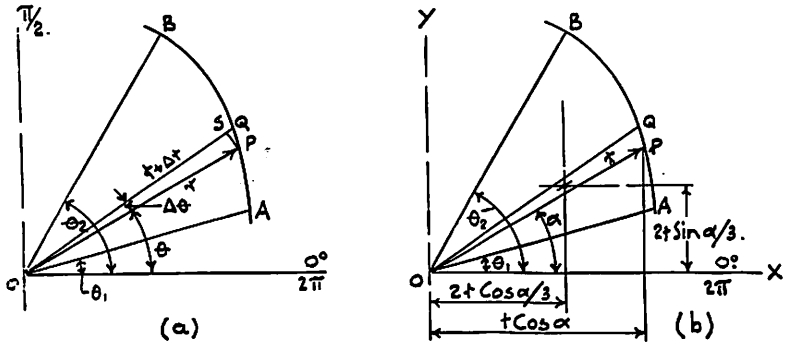

Fig. 47.

Polar Integration.

In Fig. 47 the area of the triangular element POQ is $r^2\triangle\theta/2$ plus a small triangular portion PSQ. When $\triangle\theta$ becomes infinitely small this area becomes $r^2 d\ \theta/2$. The area of the whole figure AOB is $\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\ \theta$. When r is unknown in terms of θ, the area can be evaluated using either Simpson's 1st or 2nd rule by dividing the angle AOB into a suitable number of equal smaller angles θ, measuring the radius vectors given thereby, and finally putting the squares of these through the appropriate multipliers and multiplying the sum by $\frac{1}{2}\ \theta/3$ or $\frac{1}{2}\ 3\ \theta/8$, depending on the rule used and remembering that θ is measured in radians.

The foregoing process for finding the area of figures bounded by two inclined straight lines and a third curved line is important. An example selected to show the method and at the same time investigate its accuracy will not be out of place. For this purpose the function $r = 18\ \theta/\pi$ has been slected. The curve is a form of the Spiral of Archimedes ($r = a\ \theta$); it has radius vectors which are integers at $\pi/18$ (= 10°), $\pi/9$ (= 20°), etc., and is shown in Fig. 48.

That part which is shaded has an area given exactly by :—

$$A = \int_0^{\pi/2} r^2 d\ \theta/2 = \tfrac{1}{2} \int_0^{\pi/2} (18/\pi)^2 \theta^2\ d\ \theta$$

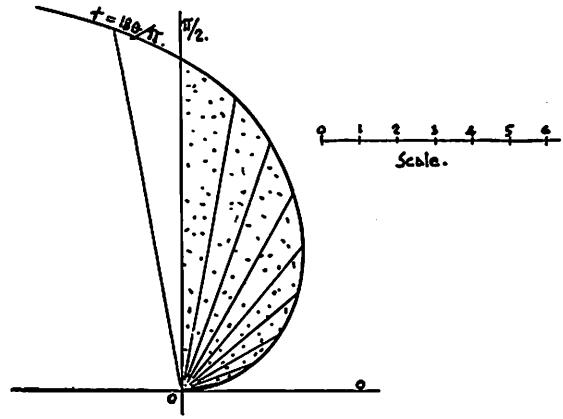

Fig. 48.

$$= \left[\frac{1}{2}\left(\frac{18}{\pi}\right)^2 \frac{\theta^3}{3}\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{18}{\pi}\right)^2 \frac{\pi^3}{24} = 21{\cdot}2 \text{ sq. units.}$$

In the approximate method, $\pi/2$ is divided into 9 equal angles, each $\pi/18$ radians. The radius vectors are shown in Fig. 48 and the area calculation is arranged as follows :—.

Angle		Rad. Vec. r	r^2	S.M.	Products
0		0	0	1	0
$\pi/18$	$(=10°)$	1	1	3	3
$\pi/9$	20°	2	4	3	12
$\pi/6$	30°	3	9	2	18
	40°	4	16	3	48
	50°	5	25	3	75
	60°	6	36	2	72
	70°	7	49	3	147
	80°	8	64	3	192
$\pi/2$	90°	9	81	1	81
				Σ =	648

$$\text{Area} = \frac{1}{2} 648 \frac{\pi}{18} \frac{3}{8} = 21{\cdot}2 \text{ sq. units.}*$$

* The curve considered is a special case of the function $r = a\,\theta + b$. The area of a sector is $\int_0^{\theta} (a^2\,\theta^2 + 2\,a\,b\,\theta + b^2)\,d\,\theta$; this is analogous to $\int_0^{l} (ax^2 + bx + c)\,dx$ considered in finding (*q.v.*) the multipliers for Simpson's 1st Rule, which, therefore, correctly integrates the curve $r = a\,\theta$, and the area obtained would be exact and not merely an approximation.

This result agrees with that previously obtained and therefore the method can be used with confidence.

Centroids of Sectors.

The centroid of AOB, Fig. 47, is sometimes required. The elementary triangle has its centroid at a distance $2r/3$ from the origin, and therefore the distances of this point from the OX and OY axis are $2r \sin \alpha/3$ and $2r \cos \alpha/3$. The moments of the elementary triangle about OX and OY are—

$$\tfrac{1}{2} r^2 d\theta \times 2r \sin \alpha/3 \text{ and } \tfrac{1}{2} r^2 d\theta \times 2r \cos \alpha/3.$$

The moments of the complete sector AOB are—

$$\text{About OX} \quad \int_{\theta_1}^{\theta_2} \tfrac{1}{3} r^3 \sin \alpha \, d\theta$$

$$\text{About OY} \quad \int_{\theta_1}^{\theta_2} \tfrac{1}{3} r^3 \cos \alpha \, d\theta$$

and these may be evaluated approximately, using Simpson's rules. The position of the centroid would be found by dividing the moment by the area, and a typical calculation is given below Fig. 49.

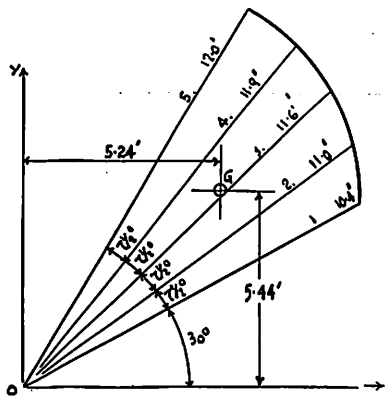

Fig. 49.

Calculation for Fig. 49.

Vector	Length r	r^2	S.M.	F(A)
1	10·4′	108	1	108
2	11·0′	121	4	484
3	11·6′	135	2	270
4	11·9′	142	4	568
5	12·0′	144	1	144
				1574

$\theta = 7\tfrac{1}{2}°$
$= 0{\cdot}131$ rad.

$A = \tfrac{1}{2}\, 1574 \dfrac{0{\cdot}131}{3}$

$= 34{\cdot}3$ sq. ft.

Vector	r^3	Cos α	r^3 cos α	S.M.	F (M_x)
1	1125	0·866	974	1	974
2	1331	0·793	1055	4	4220
3	1561	0·707	1104	2	2208
4	1685	0·609	1026	4	4104
5	1728	0·500	864	1	864
					12370

$$M_x = \tfrac{1}{3}\ 12370\ \frac{0{\cdot}131}{3} = 179{\cdot}8 \text{ ft.}^3 \text{ units.}$$

$$G_x = 179{\cdot}8/34{\cdot}3 = 5{\cdot}24 \text{ ft.}$$

Vector	r^3	Sin α	r^3 Sin α	S.M.	F (M_y)
1	1125	0·500	562	1	562
2	1331	0·609	811	4	3244
3	1561	0·707	1104	2	2208
4	1685	0·793	1336	4	5344
5	1728	0·866	1496	1	1496
					12854

$$M_y = \tfrac{1}{3}\ 12854\ \frac{0{\cdot}131}{3} = 186{\cdot}8 \text{ ft.}^3 \text{ units}$$

$$G_y = 186{\cdot}8/34{\cdot}3 = 5.44 \text{ ft.}$$

Mechanical Integration.

There are three types of mechanical integrators—planimeters, integrators and integraphs—which will evaluate a definite integral, and these instruments are of great use to civil and mechanical engineers and naval architects. A definite integral can be represented by the area of a figure properly drawn to scale, and these instruments are correctly called integrators because they can measure the areas of these figures, and therefore the integrals they represent.

The Planimeter.

It is recorded that the first planimeters were devised in 1814 by J. M. Hermann, a Bavarian engineer, and by Oppikoffer of Berne in 1835. These instruments appear to have been somewhat crude and costly. The Swiss mathematician, Jacob Amsler (1854), is generally attributed with the invention of the polar planimeter and the probability is, that he succeeded in remedying the previous defects. A much improved instrument at the time was exhibited at the Paris Exposition of 1867 by Amsler Laffon of Schoffhausen. In the last 35 years, design and general finish has been much improved, and in this respect the pole wagon planimeter which does

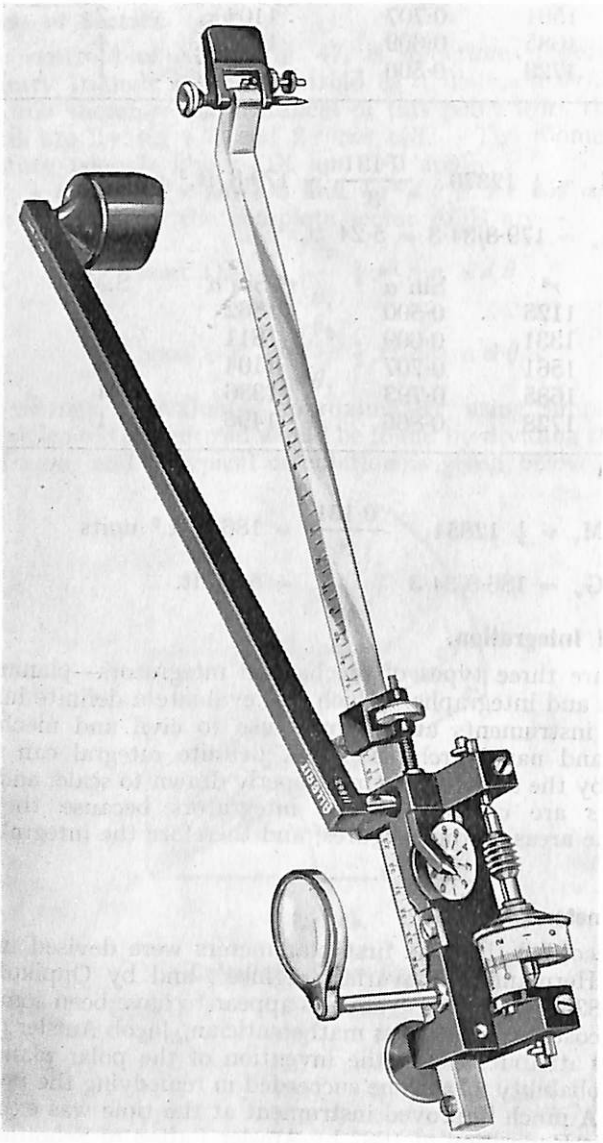

Fig. 50—Planimeter.

not have the disadvantage of being anchored; can measure very long if somewhat narrow figures. Special planimeters are also made to measure the circular charts of recording instruments.

A planimeter consists essentially of a bar A, one end having a weight, or alternatively, a pin at O, which is pressed into and anchors the planimeter to the surface on which it rests; the other end is connected to a sleeve C which carries the recording mechanism D, and also allows the tracing arm B to be varied in length and clamped. In use, the blunt pointer P on the free end of the tracing arm is traced in the clockwise direction round the figure being

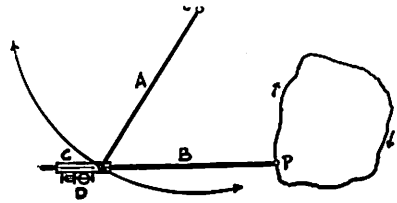

Fig. 51.

measured; this operation commences and finishes at the same point on the outline. The recording mechanism consists of a drum divided into 100 parts and reading by vernier to 1/1000 part of its circumference. The number of complete revolutions is recorded on a horizontal disc by means of an endless screw working in a wheel with ten teeth. The circumference of the roller and the length of the tracing arm determine the proportional scales of the instrument, and numbers are engraved on the tracing arm which are units of the areas being measured when the anchor point O is *within* the figure being measured. The fixed type of planimeter measured in sq. inches. The proportional type must be set to the scale and instructions for this adjustment are supplied with each instrument.

In use a reading is taken at the recording wheels. The outline of the figure is traversed by the tracing point and a second reading taken. The area is then calculated from these readings in a manner which depends on whether the anchor point O is outside or inside the figure.

If the scale of a drawing is $1/n$ inch = 1 foot, then every square inch of actual area represents n^2 square feet. Therefore the actual area in square inches which is computed by the planimeter must be multiplied by *the square of the reciprocal of the scale* to obtain the true area, thus, if the scale is 3/4 in. = 1 foot, the multiplier is $(4/3)^2 = 16/9$.

Planimeters in use should be carefully handled. The greatest cause of inaccuracy occurs when the tracing arm and the centre line of the recording mechanism are not parallel as intended by the makers.

Theory of Planimeter.

Since the arm and roller are truly parallel, the roller will not revolve if the arm is dragged in its own line, but should the arm be moved at right angles to its length, the circumference of the roller would measure the distance moved. If now the arm be moved obliquely the roller would slip the longitudinal, and roll the amount of transverse motion.

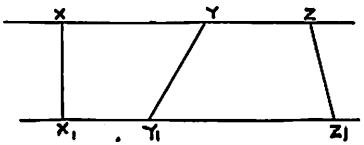

Fig. 52.

In Fig. 52, X Y Z and $X_1Y_1Z_1$ are two parallel positions of the roller. A point on the circumference always measures $X X_1$ irrespective of whether it moved along $Y Y_1$ or $Z Z_1$ and this distance $X X_1$ is that whereby the area is extracted.

In Fig. 53 (a) the movement PQ_1R must be considered infinitely small and therefore coincide with PQR.

The difference of the readings of wheel W is ds.

$$ds = k - p\,d\theta. \qquad k = ds + p\,d\,\theta.$$

$$\text{Area } APQ_1RA_1 = \text{Area } APRA_1 = lk + \tfrac{1}{2}l^2\,d\,\theta$$
$$= l\,d\,s + l\,p\,d\,\theta + l^2 d\,\theta/2$$
$$= l\,d\,s + \{2\,l\,p + l^2\}\,d\,\theta/2.$$

If P circumscribes a complete area, the total angle becomes zero, $\{2\,l\,p + l^2\}\,d\,\theta/2$ is zero and disappears.

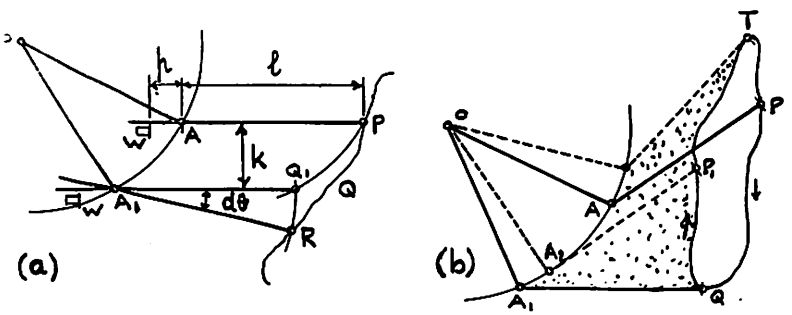

Fig. 53.

Therefore, the total area $= l\,S$, where S is measured by the wheel W, and is the difference of the final and initial readings. The area is independent of p and therefore the wheel W may be placed at any position on the tracing arm.

At (b) in Fig. 53, the tracing point commences at P and when

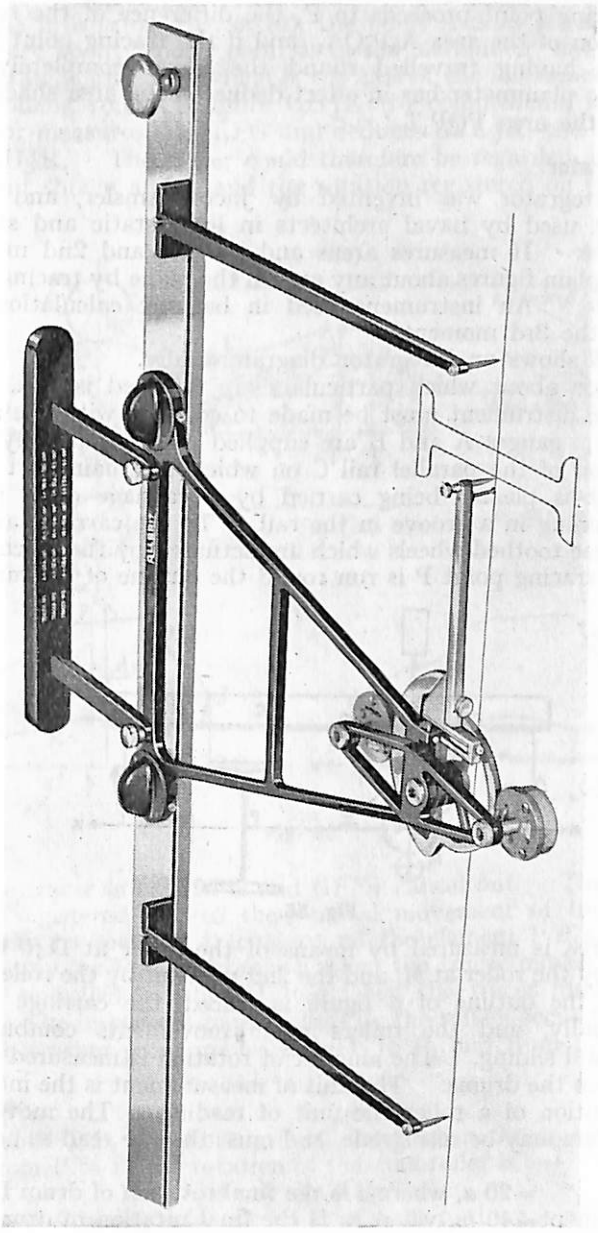

Fig. 54—Integrator.

it is at Q the area $APQA_1$ is a function of the reading on wheel W. If the tracing point proceeds to P_1 the difference of the readings is a function of the area $A_2P_1QA_1$, and if the tracing point is considered as having travelled round the figure completely from P to P, the planimeter has in effect deducted the area shaded and measured the area PQP_1T.

The Integrator.

The integrator was invented by Jacob Amsler, and is the instrument used by naval architects in hydrostatic and stability calculations. It measures areas and the 1st and 2nd moments of area of plain figures about any axis in the plane by tracing round the outline. An instrument used in ballistic calculations also measures the 3rd moment.

Fig. 55 shows an integrator diagrammically.

The axis about which particulars are required is XX. The axis of the instrument must be made to coincide with the axis of the figure; gauges A and B are supplied to effect this by fixing the position of the parallel rail C on which the main part of the instrument is placed, being carried by a carriage on a pair of wheels running in a groove in the rail. To the carriage are connected three toothed wheels which are actuated by the tracing arm when the tracing point P is run round the outline of a figure.

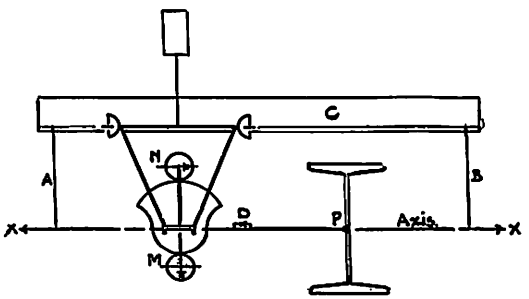

Fig. 55.

The area is measured by means of the roller at D; the 1st moment by the roller at M, and the 2nd moment by the roller at N.

When the outline of a figure is traced, the carriage moves longitudinally, and the rollers make movements combined of rotation and sliding. The amount of rotation is measured by the divisions on the drums. The unit of measurement is the inch, and one revolution of a roller the unit of reading. The movements of the rollers may be retrograde, and must then be read as negative numbers.

Area $= 20\,a$, where a is the final rotation of drum D.

1st Moment $= 40\,m$, where m is the final rotation of drum M.

2nd Moment $= 320\,a - 100\,i$, where i is the final rotation of drum N.

Theory of Integrator.

In the integrator, Fig. 56 (a), the point A does not swing in a circular arc about a polar point as in the planimeter, but oscillates in the straight line XX, which is the axis of the instrument.

In tracing round a figure GHJK (b) commencing at G, the integrator measures GHJEFG and deducts GFEJK, the difference being GHJK. The tracer could therefore be regarded as making the circuit shown at (c), and the rotation registered on the rollers

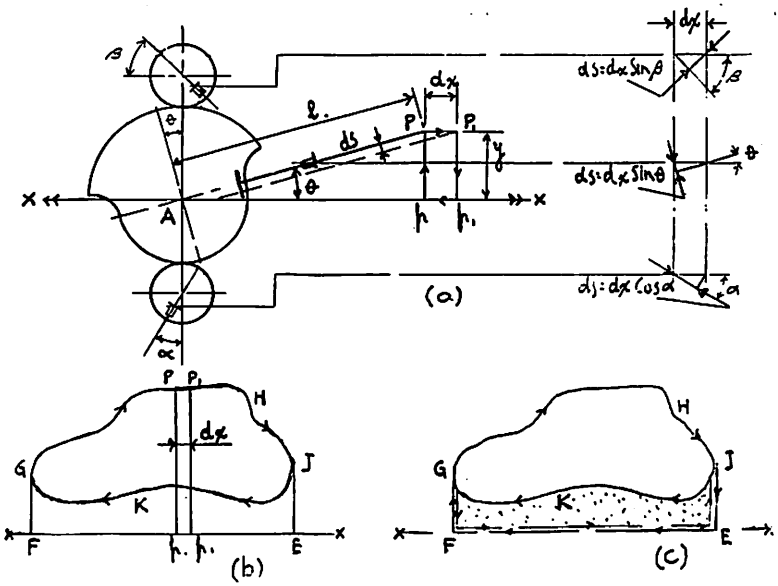

Fig. 56.

when the tracer is on JEFG and GFEJ cancel out. There is no rotation registered due to the vertical movement of the tracer, which may be seen by tracing round the element P $P_1 p_1 p$, commencing, say, at P ; the rotation when the tracer moves from P_1 to p_1 is cancelled by the rotation when the tracer moves from p to P. The important registrations by the rollers occur therefore when the tracing point moves from P to P_1 and from p_1 to p.

The Area.

The shift P P_1 is infinitely small, and when the tracing point moves from P to P_1 the rotation of the area roller is :—

$$ds = \sin \theta \, dx$$

When the tracing point moves from p_1 to p, there is no rotation of the roller.

Hence $d s = \sin \theta \, dx = y \, dx/l$

$l \, d s = y \, dx =$ Area of Element.

If s is the final rotation, $l s =$ area of figure.

Let r be the radius of roller = 0·398 ins. and the length of tracing arm, l = 8·0 ins., then $l \times s$ (1 rev.) = $8 \times 2\pi \times 0{\cdot}398$ = 20 sq. ins. and for a revs, area = 20 a sq. ins.

The 1st Moment.

In Fig. 56 the tracer arm is shown turned through an angle θ, and the moment wheel through an angle α. If the ratio of the radii of the wheels is n,

$$\alpha = -n\theta = n\theta \text{ numerically.}$$

So far as the element $y \, dx$ is concerned, the moment roller measures $dx \cos \alpha$ when the tracer moves from P to P_1, and $-dx$ when it moves from p_1 to p.

Hence for 1st moment roller—

$$\begin{aligned} d s &= dx \cos \alpha - dx \\ &= dx \, (\cos n \, \theta - 1) \end{aligned}$$

n is generally equal to 2, and

$$\begin{aligned} d s &= (\cos 2 \, \theta - 1) \, dx \\ &= -2 \sin^2 \theta \, dx \\ &= 2 \, y^2 \, dx/l^2 \text{ numerically.} \\ &= \frac{4}{l^2} \, \frac{y}{2} \, y \, dx \\ &= \frac{4}{l^2} \, \frac{y}{2} \text{ (Area of Element)} \\ &= \frac{4}{l^2} \text{ (Moment of Element)} \\ &= \frac{4}{l^2} \, d \, M_1 \end{aligned}$$

If S is the final rotation of the roller—

$$S = \frac{4}{l^2} M_1$$

$$M_1 = l^2 \, S/4, \text{ about XX.}$$

Let r, the radius of roller be 0·398 ins., and l = 8 ins., then :—

$$\begin{aligned} M_1 \text{ (one rev.)} &= 64 \times 2\pi \times 0{\cdot}398/4 \\ &= 40 \text{ in.}^3 \text{ units.} \end{aligned}$$

and for m revs.,

$$\text{1st Moment} = 40 \, m \text{ in.}^3 \text{ units.}$$

The 2nd Moment.

The ratio of the radii of the toothed wheels is k, and $\beta = -k\theta = k\theta$, numerically.

The rotation made by the roller when tracer moves from P to P_1 is $dx \sin \beta$, and from p_1 to p, the rotation is zero.

Hence, if ds is the rotation for the element $y\,dx$,

$$ds = dx \sin \beta$$

The ratio k is 3, therefore $\beta = 3\theta$.

$$ds = dx \sin 3\theta$$

$$= (3 \sin \theta - 4 \sin^3 \theta)\, dx$$

$$= 3 \frac{y}{l} dx - 4 \frac{y^3}{l^3} dx$$

$$= \frac{3}{l}(y\,dx) - \frac{12}{l^3}\left(\frac{y^3 dx}{3}\right)$$

$$\frac{3}{l}(\text{area of element}) - \frac{12}{l^3}(\text{2nd moment of element})$$

$$\frac{l^3\,ds}{12} = \frac{l^2}{4}(\text{area of element}) - d\,M_2$$

$$d\,M_2 = \frac{l^2}{4}(\text{area of element}) - \frac{l^3\,ds}{12}$$

Hence the 2nd Moment of figure about XX is $M_2 = l^2/4$ (area of figure) $- l^3\,s/12$, and when $l = 8$ ins., and r the radius of roller is made 0·373 ins., the 2nd Moment for i revolutions of the 2nd Moment roller is :—

$$M_2 = \frac{64}{4} 20\,a - \frac{512}{12} \times 2\,\pi \times 0{\cdot}371\,i$$

$$= 320\,a - 100\,i$$

The Integraph.

The integraph was invented by a Russian engineer, M. Abdank-Abakanowicz, whose work entitled *Les Intégraphs : la courbe intégrale et ses applications* (Paris, Gauthier-Villars), or the German translation by *Bitterli* (Leipzig, Teubner), should be consulted by anyone requiring more information on this subject than can be given here. The instrument is superior to the integrator inasmuch as it graphs the integral curve of a fundamental curve mechanically.

The mechanism illustrated in Fig. 58 is so contrived that, in tracing along a fundamental curve OP with tracing point P, a second point P_1 graphs the integral curve O_1P_1.

The instrument consists of an axle about 24 inches long, with heavy wheels and prior to being used is moved about until the

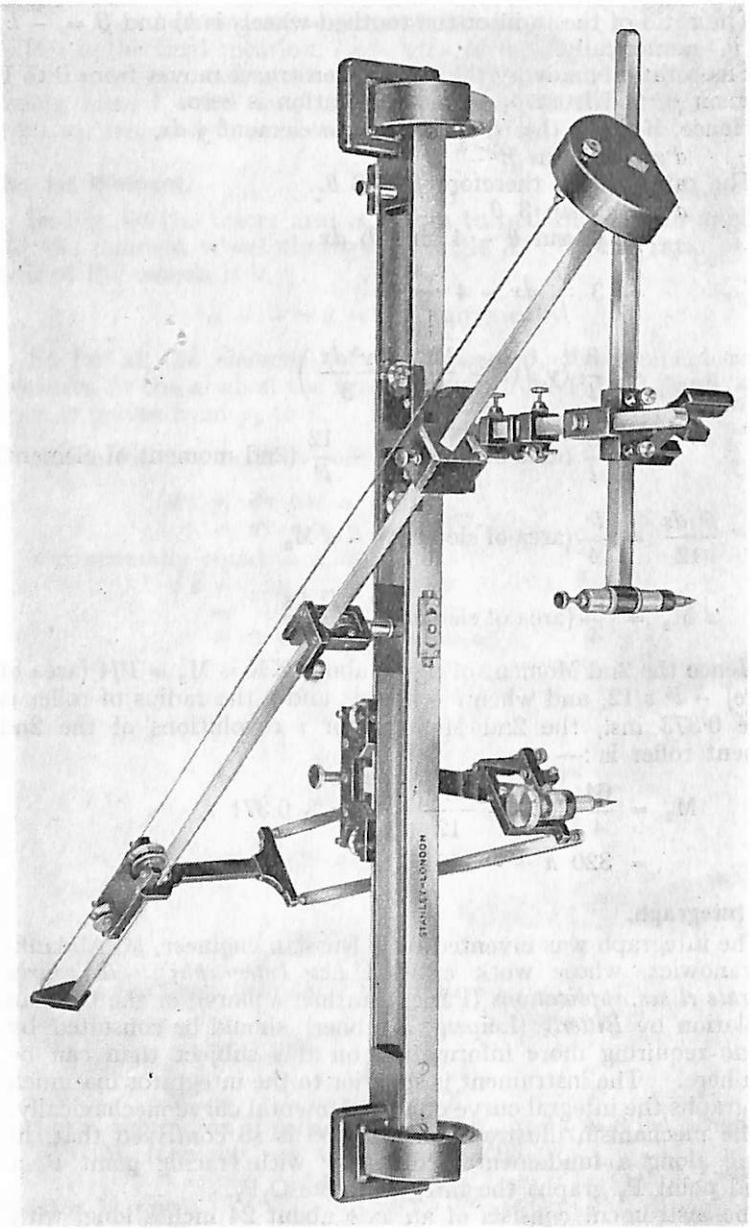

Fig. 57—Integraph.

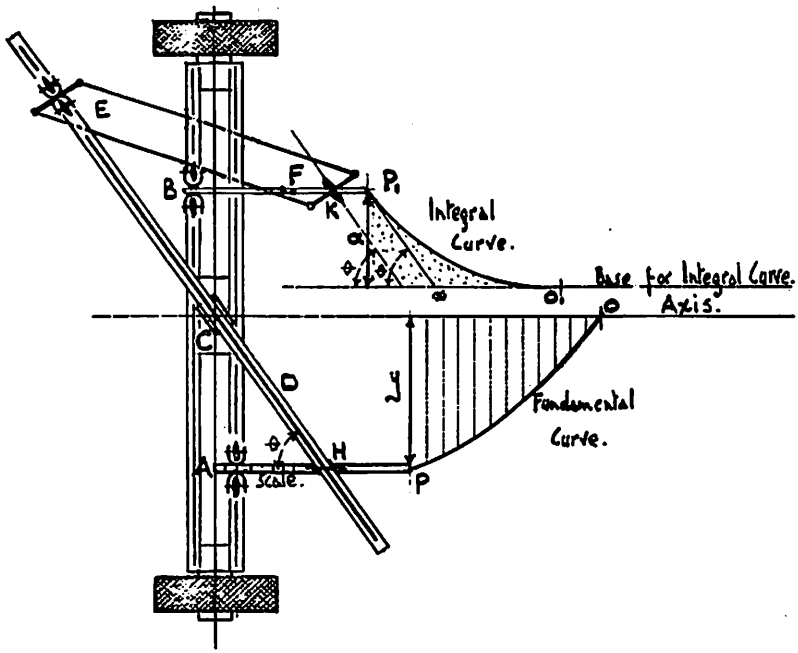

Fig. 58.

centre C coincides with the axis of the figure being investigated. Carriages A and B run on wheels in grooves cut in the axle, and from these carriages arms AP and BP_1 extend at right angles. On AP is a scale with vernier, in inches, and zero of this scale is at the centre line of the axle. The long radial arm D can be clamped at any position H on the scale AP, but the fitting allows rotation of the radial arm, which is also the case at the pivoted block C where, however, the radial arm can also slide easily through a sleeve in the block. P_1 at the end of arm BP_1 is the point at which the integral curve is graphed. K is a small wheel having a sharp edge which rests on the paper, and can rotate about a vertical axis through its centre. This wheel always lies parallel to the radial arm D by virtue of the parallel motion EF. End E of the parallel motion travels on a carriage on the radial arm. P_1 therefore records the movement of wheel K.

If P is kept in position and the machine rolled on its main wheels, the radial arm retains its position and the wheel K moves in the direction of its circumference, that is, in a direction parallel to the radial arm. If, however, P is moved laterally during the movement, the radial arm will rotate about C, and the wheel K at every instant will be parallel to the radial arm; therefore the tangent of the integral curve being graphed will be parallel to the line of wheel K.

If θ is the angle of inclination of the tangent at P_1

$$\text{then } \frac{da}{dx} = \tan\ \theta = AC/AH$$

If m is the scale, then AH is equal to m and—

$$AC = m\frac{da}{dx} = y$$

$$da = \frac{1}{m}\, y\, dx$$

$$a = \frac{1}{m}\int y \,.\, dx$$

The ordinate of the fundamental curve y is proportional at any instant to the differential coefficient of the curve being graphed, the ordinate of which is therefore a measure of the integral (or area) of the fundamental curve.

In setting up the instrument for use, the radial arm is kept at right angles to the axle ; the point C and therefore P is made to travel directly over the axis of the curve being investigated. The wheel K is lowered to the paper at some convenient distance from the axis and the instrument is again rolled along. P_1 then graphs a line parallel to the axis, and this is the base line for the integral curve. The tracing point P is then brought back to O, and when it is made to trace over the curve OP the corresponding integral curve O_1P_1 is graphed by the recording pen.

ACKNOWLEDGMENT.

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BIBLIOGRAPHY.

The booklet entitled *The Stanley Integraph : Theory and Practice*, by S. Kirkby, B.Sc., published by Messrs. Stanley, price 5/- at the present time, is an excellent treatise on the integraph.

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